Unit 83.3
Continuation of Unit 83.1
Number theoretic finger exercises to train creative thinking

83.3.1
Check the suitability of the polynomials

$$p(n,A,x) = A+\begin{vmatrix} x^1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -x^2 & 1 & 2 & 0 & 0 & 0 & 0 & \ddots \\ x^3 & 1 & 3 & 3 & 0 & 0 & 0 & \ddots \\ -x^4 & 1 & 4 & 6 & 4 & 0 & 0 & \ddots \\ x^5 & 1 & 5 & 10 & 10 & 5 & 0 & \ddots \\ -x^6 & 1 & 6 & 15 & 20 & 15 & 6 & \ddots \\ x^7 & 1 & 7 & 21 & 35 & 35 & 21 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

n by n determinant

$$n,A,x\in\mathbb{N} \qquad (1)$$

as prim-generating polynomials.

83.3.2

Lateral to equ. (1)

$$\frac{p(n,0,x)}{p(n,0,1)} = \sum_{i=1}^{x} i^{n-1} \qquad (2)$$

$$p(n,0,1) = n! \qquad (3)$$

determinant representation of n!,
corresponding to Pascal's triangle (see Unit 2, example 3; Unit 6, Unit 42)

83.3.3

Lateral to equ. (1)

$$\begin{vmatrix} x^1 & 1 & 0 & 0 \\ -x^2 & 1 & 2 & 0 \\ x^3 & 1 & 3 & 3 \\ -x^4 & 1 & 4 & 6 \end{vmatrix} = 6 \begin{vmatrix} x^1 & 1 \\ -x^2 & 1 \end{vmatrix}^2 \qquad (4)$$

Find more!

83.3.4

Find polynomials p(n,A,x) which generate as many primes as possible, esp. for small x,
f.i. p(2,17,x), p(2,41,x), …
Example, screening

x	p(2,17,x)	p(3,17,x)	p(4,17,x)
0	17p	17p	17p
1	19p	23p	41p
2	23p	47p	233p
3	29p	101p	881p
4	37p	197p	2417p
5	47p	347p	5417p
6	59p	563p	10601p
7	73p	857p	18833
8	89p	1241	31121p
9	107p	1727	48617
10	127p	2327	72617p
11	149p	3053	104561p
12	173p	3917p	146033p
13	199p	4931p	198761p
14	227p	6107	264617
15	257p	7457p	345617
16	289	8993	443921

17	323	10727	561833
18	359p	12671p	701801
19	397p	14837	866417p
20	437	17237	1058417
21	479p	19883	
22	523p	22787p	1536233
23	569p	25961	
24	617p	29417	2160017p
25	667	33167	2535017p
26	719p	37223p	2956841p
27	773p	41597p	
28	829p	46301p	3956081p
29	887p	51347p	
30	947p	56747p	
31	1009p	62513	
32	1073	68657	
33	1139	75191	
34	1207	82127	

35	1277p	89477p	9525617
36	1349	97253	
37	1423p	105467p	11861033p
38	1499	114131	13177961p
39	1577	123257	
40	1657p	132857p	16137617p
41	1739		17791721p
43	1909no	164621p	
44	1997p	176237p	
45	2087p	188387	
46	2179p	201083	
47	2273p	214337	
48	2369	228161	
49	2467p	242567	
50	2567	257567	
51	2669	273173	
52	2773no	289397p	

53	2879p	306251	49146281p
54	2987	323747	
55	3097	341897	
56	3209p	360713	
57	3323p	380207p	
58	3439no	400391p	
59	3557p	421277	
60	3677p	442877	

Continue and discuss, esp. the conjecture:
n=0(1)60
** if p(2,A,x) not prime then p(3,A,x) almost[1)] (probability = ?) not prime,
** if p(2,A,x) prime then p(3,A,x) frequently (probability = ?) prime.
** p(2,A,x) = prime → guide to detect larger primes?

83.3.4
Find a recurrence formula and 2by2 matrix chain product representations for p(n,0,x) (see [1] and Unit 7).

83.3.5 (see Unit 83.1)
p(n,A,x) generates for x = 0(1)a b primes.

c_1 of these have the final digit 1,
c_3 3,
c_7 7,
c_9 9.
Calculate and discuss b(n,A,a) and the frequencies
$\frac{c_1}{b}(n, A,a)$, $\frac{c_3}{b}(n,A,a)$, $\frac{c_7}{b}(n,A,a)$, $\frac{c_9}{b}(n,A,a)$
for representative values of n, A and a.
Find associated approximation formulas for these functions.
83.3.6
Check the suitability of the polynomials q(n,A,x) = A + orthogonal polynomial [2,3] or other special polynomial (see Units 6, 7, 12, and 42) as prime generator.
Lateral
Find the corresponding determinant and 2by2 matrix chain product representations for q(n,0,x).
Casting show
83.3.6L.1, see Unit 12

$$D_0(x) = 1$$

$$D_n(x) = \begin{vmatrix} x & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & x & 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & x & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & x & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & x & 1 & \ddots \\ 0 & 0 & 0 & 0 & 1 & x & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{vmatrix}$$

n by n determinant

83.3.6L.2 Tschebysheff polynomials, first kind

$$T_0(x) = 1$$

$$T_n(x) = \begin{vmatrix} x & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2x & 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 2x & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 2x & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 2x & 1 & \ddots \\ 0 & 0 & 0 & 0 & 1 & 2x & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{vmatrix}$$

n by n determinant

83.3.6L.3 Tschebysheff polynomials, second

kind

$$U_0(x) = 1$$

$$U_n(x) = \begin{vmatrix} 2x & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2x & 1 & 0 & 0 & 0 & \ddots \\ 0 & 1 & 2x & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 2x & 1 & 0 & \ddots \\ 0 & 0 & 0 & 1 & 2x & 1 & \ddots \\ 0 & 0 & 0 & 0 & 1 & 2x & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{vmatrix}$$

n by n determinant

83.3.6L.4 Hermite polynomials

$$H_0(x) = 1$$

$$H_n(x) = \begin{vmatrix} 2x & 2 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2x & 2 & 0 & 0 & 0 & \ddots \\ 0 & 2 & 2x & 2 & 0 & 0 & \ddots \\ 0 & 0 & 3 & 2x & 2 & 0 & \ddots \\ 0 & 0 & 0 & 4 & 2x & 2 & \ddots \\ 0 & 0 & 0 & 0 & 5 & 2x & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{vmatrix}$$

n by n determinant

83.3.6L.5 Polynomials represented by determinants in Units 6, 42, and 61

83.3.7 Examples to 83.3.5 calculated by Dr. Dagmar Schönfeld

83.3.7.1.1

$$b(2,17,10^4) = 1649,\ b(2,17,5*10^4) = 6967,$$

$$b(2,17,10^5) = 13104$$

$$\frac{c_1(2,17,\infty)}{b} = \frac{1}{5},\ \frac{c_3(2,17,\infty)}{b} = \frac{1}{5},\ \frac{c_7(2,17,\infty)}{b} = \frac{3}{5},$$

$$\frac{c_9(2,17,\infty)}{b} = 0$$

83.3.7.1.2

$$b(3,17,10^4) = 1518,\ b(3,17,5*10^4) = 6685,$$

$$b(3,17,10^5) = 12621$$

$$\frac{c_1(3,17,\infty)}{b} = \frac{2}{5},\ \frac{c_3(3,17,\infty)}{b} = \frac{1}{5},\ \frac{c_7(3,17,\infty)}{b} = \frac{2}{5},$$

$$\frac{c_9(3,17,\infty)}{b} = 0$$

83.3.7.1.3

$$b(4,17,10^4) = 680,\ b(4,17,5*10^4) = 2755,$$

$$b(4,17,10^5) = 5150$$

$$\frac{c_1(4,17,\infty)}{b} = 0,\ \frac{c_3(4,17,\infty)}{b} = 0,\ \frac{c_7(4,17,\infty)}{b} = 1,$$

$$\frac{c_9(4,17,\infty)}{b}=0$$

83.3.7.2.1

$$b(2,41,10^5)=13754$$

$$\frac{c_1(2,41,\infty)}{b}=0,\ \frac{c_3(2,41,\infty)}{b}=\frac{3}{5},\ \frac{c_7(2,41,\infty)}{b}=\frac{1}{5},$$

$$\frac{c_9(2,41,\infty)}{b}=\frac{1}{5}$$

83.3.7.2.2

$$b(3,41,10^5)=5811$$

$$\frac{c_1(3,41,\infty)}{b}=\frac{2}{3},\ \frac{c_3(3,41,\infty)}{b}=0,\ \frac{c_7(3,41,\infty)}{b}=\frac{1}{3},$$

$$\frac{c_9(3,41,\infty)}{b}=0$$

83.3.7.2.3

$$b(4,41,10^5)=7604$$

$$\frac{c_1(4,41,\infty)}{b}=1,\ \frac{c_3(4,41,\infty)}{b}=0,\ \frac{c_7(4,41,\infty)}{b}=0,$$

$$\frac{c_9(4,41,\infty)}{b} = 0$$

Discussion.
Find the general solution.

References

[1] D. Zwillinger (ed). CRC Standard Mathematical Tables and Formulae. 32nd ed., part 6.13
ISBN 978-1-4398-3548-7

[2] http://mathworld.wolfram.com/OrthogonalPolynomials.html

[3] https://en.wikipedia.org/wiki/Orthogonal_polynomials

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1) ** See almost integers
http://mathworld.wolfram.com/AlmostInteger.html
https://en.wikipedia.org/wiki/Almost_integer

** See Unit 83.4