Unit n°68

E to the Pi, recursively

$$e^{\pi} = 23.140692632779269005729086367 9\ldots$$

$$= \lim_{n\to\infty} \sqrt[2^n]{4x(n+1)}$$

$$= \lim_{n\to\infty} \sqrt[2^{n-1}]{\frac{x(n+1)}{x(n)}} \; ;$$

$$x(n+1) = \left(x(n) + \sqrt{x(n)^2 - 1}\right)^2 \; ; n=0,1,2,3,...$$

$$x(0) = \sqrt{2}$$

$$e^{\pi}(n=1) = 4x(1) = (\sqrt{2}+1)^2 = 4\frac{\sqrt[2]{2}+1}{\sqrt[2]{2}-1} = 23.3137085...$$

$$e^{\pi}(n=2) = \sqrt{4x(2)} = 2\frac{\sqrt[4]{2}+1}{\sqrt[4]{2}-1} = 23.14085407...$$

$$e^{\pi}(n=3) = \sqrt[4]{4x(3)} = ??????? = 23.1406926 7...$$

A close relative

$$i^i = \frac{1}{e^{\frac{\pi}{2}}} = 0,207879576... \; ; \; i = \sqrt{-1}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt[2^n]{4x(n)}}$$

$$= \lim_{n\to\infty} \sqrt[2^n]{\frac{x(n)}{x(n+1)}}$$

An other approach

$$e^{\pi} = 23.14069263277926...$$

$$= \lim_{n\to\infty} \left(\frac{\tan\frac{\alpha(n)}{2}}{2} \right)^{-2^{-(n-1)}}$$

$$\sin\alpha(n+1) = \tan^2\frac{\alpha(n)}{2} \; ; \; n=0,1,2,3,...$$

$$\alpha(0) = \frac{\pi}{4}$$

** **What does give this recurrence equation for other initial values $\alpha(0)$?**

** **How about the recurrence equation**

$$\tan\alpha(n+1) = \tan^2 \frac{\alpha(n)}{2} \quad ; n=0,1,2,3,...$$

$$\alpha(0) = \frac{\pi}{4} \quad \text{or ...} \quad ?$$

** **Find other recurrence equations for e^{π}. Discussion!**