Unit 555.3.1
Supplement 1 to
Unit 555.1.corr.corr and
Unit 555.2
Comparision of e and π representations

$$\sum_{i=1}^{\infty} \frac{1}{i!} = e - 1$$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \qquad (1.1,2)$$

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Open problems

$$\sum_{i=1}^{\infty} \frac{1}{(i!)^2} = ??? \qquad (2.1)$$

1.27958530233606726743720444081153335328584110278545905407084…

$$\sum_{i=1}^{\infty} \frac{1}{(i!)*(i^2)} = ??? \qquad (2.2)$$

1.14649907252864280790119520246471868861929177338810947043035…

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e-sandbox

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$$e-1=1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cfrac{6}{6+\cfrac{7}{7+\cdots}}}}}}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\cfrac{5}{6+\cdots}}}}}}$$

$$=[1;1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,\ldots]$$

$$=[-1;1,-1,-2,3,2,-5,-2,7,2,-9,-2,11,2,\ldots]$$

(3)

Lateral

$$\sqrt{e}-1=\cfrac{1}{1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\cfrac{10}{11+\cfrac{12}{13+\cdots}}}}}}}$$

$$=[0;1,1,1,5,1,1,9,1,1,13,1,1,17,1,1,\ldots]$$

(4)

$$e^2-1=6+\cfrac{2}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\cfrac{1}{13+\cfrac{1}{15+\cdots}}}}}}$$

$$=[6;\tfrac{5}{2},7*2,\tfrac{9}{2},11*2,\tfrac{13}{2},15*2,\tfrac{17}{2},19*2,\ldots]$$

$$=[-1;+1,-\tfrac{1}{2},-2,+\tfrac{3}{2},+2,-\tfrac{5}{2},,-2,\ldots]$$

(5)

$$1=\cfrac{2}{1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\cfrac{7}{6+\cfrac{8}{7+\cdots}}}}}}}=\cfrac{2}{1+\cfrac{4}{3+\cfrac{6}{5+\cfrac{8}{7+\cfrac{10}{9+\cfrac{12}{11+\cfrac{14}{13+\cdots}}}}}}}$$

$$= \cfrac{3}{2+\cfrac{5}{4+\cfrac{7}{6+\cfrac{9}{8+\cfrac{11}{10+\cfrac{13}{12+\cfrac{15}{14+\cdots}}}}}}} \qquad (6)$$

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$$e - 1 = 1+ \frac{1}{2}(1+ \frac{1}{3}(1+ \frac{1}{4}(1+ \frac{1}{5}(1+ \frac{1}{6}(1+ \ldots.))))) \qquad (7)$$

$$\sqrt{e} - 1 = \frac{1}{2}(1+ \frac{1}{4}(1+ \frac{1}{6}(1+ \frac{1}{8}(1+ \frac{1}{10}(1+ \ldots.))))) \qquad (8)$$

$$e^2 - 1 = \frac{2}{1}(1+ \frac{2}{2}(1+ \frac{2}{3}(1+ \frac{2}{4}(1+ \frac{2}{5}(1+ \ldots.))))) \qquad (9)$$

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$$e - 1 = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n - 1 \qquad (10)$$

$$\sqrt{e} - 1 = \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n - 1 \qquad (11)$$

$$e^2 - 1 = \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n - 1 \qquad (12)$$

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$$e - 1 = \begin{vmatrix} 0 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \rightarrow \\ -1 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \rightarrow \\ +1 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \rightarrow \\ -1 & 0 & 0 & \mathbf{0} & \mathbf{1} & \rightarrow \\ +1 & 0 & 0 & 0 & \mathbf{0} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix}$$

$$n = n \; ; \; \mathbf{n} = \frac{1}{n!} \; ; \; \mathbf{0} = \frac{1}{0!} = 1 \qquad (14)$$

$$e^x - 1 = \left| \begin{matrix} 0 & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \rightarrow \\ -x^1 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \rightarrow \\ +x^2 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \rightarrow \\ -x^3 & 0 & 0 & \mathbf{0} & \mathbf{1} & \rightarrow \\ +x^4 & 0 & 0 & 0 & \mathbf{0} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{matrix} \right| \qquad (15)$$

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Interlude 1

Transformation of the **c**ontinued **f**raction

$$cf = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5 + \cfrac{b_6}{a_6 + \cdots}}}}}} \rightarrow =$$

into the **s**imple **c**ontinued **f**raction

$$scf = A_0 + \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{A_3 + \cfrac{1}{A_4 + \cfrac{1}{A_5 + \cfrac{1}{A_6 + \cdots}}}}}}$$

$= [A_0; A_1, A_2, A_3, A_4, A_5, A_6, \ldots]$

with

$A_0 = a_0 \; ; \; A_1 = \frac{a_1}{b_1}$

$$A_{i+1} = \frac{1}{A_i} * \frac{a_i * a_{i+1}}{b_{i+1}} \; ; \; i=1,2,3,\ldots \qquad (20)$$

Example 1

$$e-1 = 1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cfrac{6}{6+\cfrac{7}{7+\cdots}}}}}} =$$

$$[1;1,\frac{2}{1},\frac{1*3}{2},\frac{2*4}{1*3},\frac{1*3*5}{2*4},\frac{2*4*6}{1*3*5},\frac{1*3*5*7}{2*4*6},\frac{2*4*6*8}{1*3*5*7},\ldots] \qquad (21)$$

$$\frac{2*4*6*8*10*12*14*16*\ldots}{1*3*5*7*9*11*13*15*\ldots} = ??? \; , \qquad (22)$$

see also [1,2].

$$\frac{2*2}{1*3} * \frac{4*4}{3*5} * \frac{6*6}{5*7} * \frac{8*8}{7*9} * \frac{10*10}{9*11} * \frac{12*12}{11*13} * \frac{14*14}{13*15} *\ldots = \frac{\pi}{2} \qquad (23)$$

Example 2

$$\frac{4}{\pi} = 1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cfrac{5^2}{11+\cfrac{6^2}{13+\cdots}}}}}} =$$

$$[1;3,5*\left(\frac{1}{2}\right)^2,7*\left(\frac{2}{1*3}\right)^2, 9*\left(\frac{1*3}{2*4}\right)^2, 11*\left(\frac{2*4}{1*3*5}\right)^2,$$

$$13*(\frac{1*3*5}{2*4*6})^2, 15*\left(\frac{2*4*6}{1*3*5*7}\right)^2, 17*(\frac{1*3*5*7}{2*4*6*8})^2, \ldots]$$

(24)

Example 3

$$\frac{12}{\pi^2} = 1+\cfrac{1^4}{3+\cfrac{2^4}{5+\cfrac{3^4}{7+\cfrac{4^4}{9+\cfrac{5^4}{11+\cfrac{6^4}{13+\cdots}}}}}} =$$

$$[1;3,5*\left(\frac{1}{2}\right)^4,7*\left(\frac{2}{1*3}\right)^4, 9*\left(\frac{1*3}{2*4}\right)^4, 11*\left(\frac{2*4}{1*3*5}\right)^4,$$

$$13*(\frac{1*3*5}{2*4*6})^4, 15*\left(\frac{2*4*6}{1*3*5*7}\right)^4, 17*(\frac{1*3*5*7}{2*4*6*8})^4, \ldots]$$

(25)

Extension?

$$\frac{f(k)}{\pi^k} = 1+\cfrac{1^{2k}}{3+\cfrac{2^{2k}}{5+\cfrac{3^{2k}}{7+\cfrac{4^{2k}}{9+\cfrac{5^{2k}}{11+\cfrac{6^{2k}}{13+\cdots}}}}}} \; ; k\in\mathbb{N} \qquad (26.1)$$

Discuss value chart and graph of f(k) and the relationship of f(k) with the Riemann Zeta function [3,4].

k	f(k)
0	1.313035286…=ctanh(1)
⅛	1.511125687…
¼	1.738485384...
½	2.298390619…

1	4
2	12
3	38.236223…
4	125.96008…
5	403.08357…
6	1275.3936…
7	4018.4089…
8	12639.613…
9	39729.315
10	124841.75

(26.2)

Example 4

$$\frac{1}{\ln 2} = 1+\cfrac{1^2}{1+\cfrac{2^2}{1+\cfrac{3^2}{1+\cfrac{4^2}{1+\cfrac{5^2}{1+\cfrac{6^2}{1+\cdots}}}}}} =$$

$$[1;1,\left(\frac{1}{2}\right)^2,\left(\frac{2}{1*3}\right)^2,\left(\frac{1*3}{2*4}\right)^2,\left(\frac{2*4}{1*3*5}\right)^2,$$

$$(\frac{1*3*5}{2*4*6})^2,\left(\frac{2*4*6}{1*3*5*7}\right)^2,(\frac{1*3*5*7}{2*4*6*8})^2,\ldots] \qquad (27)$$

Example 5

$$??? = a_0+\cfrac{1^k}{a_1+\cfrac{2^k}{a_2+\cfrac{3^k}{a_3+\cfrac{4^k}{a_4+\cfrac{5^k}{a_5+\cfrac{6^k}{a_6+\cdots}}}}}} =$$

$$[a_0;a_1,a_2*\left(\frac{1}{2}\right)^k, a_3*\left(\frac{2}{1*3}\right)^k, a_4*\left(\frac{1*3}{2*4}\right)^k,$$
$$a_5*\left(\frac{2*4}{1*3*5}\right)^k, a_6*\left(\frac{1*3*5}{2*4*6}\right)^k, a_7*\left(\frac{2*4*6}{1*3*5*7}\right)^k,$$
$$a_8*\left(\frac{1*3*5*7}{2*4*6*8}\right)^k, \ldots] \qquad (28)$$

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Interlude 2

2.1

Transformation of the continued fraction

$$cf = a_0+ \cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\cfrac{b_4}{a_4+\cfrac{b_5}{a_5+\cfrac{b_6}{a_6+\cdots}}}}}} \qquad (101.1)$$

into the nested expression

$$ne = a_0+\frac{b_1}{D_2}\left(a_2+\frac{b_2b_3D_0}{D_4}\left(a_4+\frac{b_4b_5D_2}{D_6}\left(a_6+\frac{b_6b_7D_4}{D_8}(a_8+\cdots)\right)\right)\right) =$$

$$a_0+\frac{b_1a_2}{D_2}\left(1+\frac{b_2b_3a_4D_0}{a_2D_4}\left(1+\frac{b_4b_5a_6D_2}{a_4D_6}\left(1+\frac{b_6b_7a_8D_4}{a_6D_8}(1+\cdots)\right)\right)\right) \tag{101.2}$$

with

$$D_0 = 1; \quad D_2 = \begin{vmatrix} a_1 & b_2 \\ -1 & a_2 \end{vmatrix}$$

$$D_{2i} = \begin{vmatrix} D_{2i-2} & b_{2i-2} & 0 & 0 \\ 0 & a_{2i-2} & b_{2i-1} & 0 \\ -D_{2i-4} & -1 & a_{2i-1} & b_{2i} \\ 0 & 0 & -1 & a_{2i} \end{vmatrix}; \tag{101.3}$$

i=2,3,4,…

Example 6

$$\Phi = \frac{1}{2}(1+\sqrt{5}) = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}} =$$

$$1+\frac{1}{2}\left(1+\frac{1}{5}\left(1+\frac{2}{13}\left(1+\frac{5}{34}\left(1+\frac{13}{89+\cdots}(\ldots)\right)\right)\right)\right) \tag{102}$$

Example 7

$$\sqrt{2} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}}} =$$

$$1+\frac{2}{5}\left(1+\frac{1}{29}\left(1+\frac{5}{169}\left(1+\frac{29}{985}\left(1+\frac{169}{5741+\cdots}(\ldots)\right)\right)\right)\right) \tag{103.1}$$

Lateral

$\sqrt{2}$ generates via ne equ. (103.1) a sequence of scf palindromes

$\mathrm{scf}\left(\frac{169-29}{29-5}\right)=[5;1,5]$, $\mathrm{scf}\left(\frac{985-169}{169-29}\right)=[5;1,4,1,5]$,

$\mathrm{scf}\left(\frac{5741-985}{985-169}\right)=[5;1,4,1,4,1,5]$,

$\mathrm{scf}\left(\frac{33461-5741}{5741-985}\right)=[5;1,4,1,4,1,4,1,5]$,

$\mathrm{scf}\left(\frac{195025-33461}{33461-5741}\right)=[5;1,4,1,4,1,4,1,4,1,,5]$, …

$\rightarrow \mathrm{scf}(3+2\sqrt{2})$ 103.2)

2.2

Nested expression

$$C_0+\frac{A_1}{B_1}(C_1+\frac{A_2}{B_2}(C_2+\frac{A_3}{B_3}(C_3+\frac{A_4}{B_4}(C_4+\ldots))))= \quad (104.1)$$

compression → convergence acceleration into

$$C_0+\frac{A_1}{B_1B_2}\left(\begin{vmatrix}A_2 & -B_2\\ C_1 & C_2\end{vmatrix}+\frac{A_2A_3}{B_3B_4}\left(\begin{vmatrix}A_4 & -B_4\\ C_3 & C_4\end{vmatrix}+\frac{A_4A_5}{B_5B_6}\left(\begin{vmatrix}A_6 & -B_6\\ C_5 & C_6\end{vmatrix}+\cdots\right)\right)\right) \quad (104.2)$$

Example 8

$\sqrt{2}-$ equ. (103.1),

via equ. (104.2)−compressed =

$$1+\frac{2}{5*29}\left(30+\frac{5}{169*985}\left(1014+\frac{29*169}{5741*33461}(34446+\cdots)\right)\right) \quad (105)$$

Example 9

Comparison of the sqr(2) approximations g(k) and h(k) calculated from the first k ne-elements of equs (103) and (105), respectively:

k	g(k)	h(k)
1	1	1
2	1.4	1.413793103…
3	1.413793103…	1.414213198…
4	1.414201183…	<u>1.414213562</u>…
5	1.414213198…	
6	1.414213552…	
7	<u>1.414213562</u>...	

(106)

2.3

Convergence acceleration of a sequence $\langle f(k)\rangle_{k=1}^{\infty}$ to $\langle f1(k)\rangle_{k=1}^{\infty}$ with

$$f1(k) = \frac{\begin{vmatrix} f(k) & f(k+1) \\ f(k+1) & f(k+2) \end{vmatrix}}{f(k)-2f(k+1)+f(k+2)}\ ,\ k=1,2,3,\ldots \qquad (107)$$

Example 10

$\langle f(k)\rangle_{k=1}^{\infty} = \sqrt{2}$ approximation sequence calculated from equ. (103.1) versus f1(k):

k	f(k)	f1(k) via equ. (107)
1	1	1.414285714…
2	1.4	1.414213558…

3 1.413793103… 1.414213562…
4 1.414201183…
5 1.414213198…
6 1.414213552…
7 1.414213562... (108)

Example 11
Continued fraction

$$e = 2+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cfrac{6}{6+\cdots}}}}} = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\cdots}}}}} \rightarrow \quad (109.1,2)$$

transformed via equ. (101) into the nested expression

$$\frac{e}{2} = 1+\frac{1}{3}\left(1+\frac{4}{53}\left(1+\frac{54}{2119}\left(1+\frac{2120}{148329}(1+\frac{148330}{16019531}\left(1+\frac{16019532}{2467007773}\left(1+\frac{2467007774}{\ldots}(1+\cdots)\right)\right)\right)\right)\right)\right) \quad (110.1)$$

with
4 = 1*4

54 = 3*6*(1*4-1)
2120 = 5*8*(3*6*(1*4-1)-1)
148330 = 7*10*(5*8*(3*6*(1*4-1)-1)-1)
16019532 =
9*12*(7*10*(5*8*(3*6*(1*4-1)-1)-1)-1)
2467007774 =
11*14*(9*12*(7*10*(5*8*(3*6*(1*4-1)-1)-1)-1)-1)
(110.2)

$\langle m(k)\rangle_{k=1}^{\infty} = \frac{e}{2}$ −approximation sequence calculated from the first k ne-elements of equ. (110) versus m1(k)

k	m(k)	m1(k) via equ. (107)
1	1	
2	1.3per	
3	1.358490566…	
4	1.359131666…	
5	1.359140829…	
6	1.359140914…	(110.3)

Lateral
Change of sign

$$1+\frac{1}{3}\left(1+\frac{4}{89}\left(1+\frac{90}{3639}\left(1+\frac{3640}{254869}(1+\frac{254870}{27526067}\left(1+\frac{27526068}{4239014625}\left(1+\frac{4239014626}{\ldots}(1+\cdots)\right)\right)\right)\right)\right)\right) = 1{,}348690465\ldots = ??$$

(110.4)

4 = 1*4
90 = 3*6*(1*4+1)
3640 = 5*8*(3*6*(1*4+1)+1)
254870 = 7*10*(5*8*(3*6*(1*4+1)+1)+1)
27526068 =
9*12*(7*10*(5*8*(3*6*(1*4+1)+1)+1)+1)
4239014626 =
11*14*(9*12*(7*10*(5*8*(3*6*(1*4+1)+1)+1)+1)+1)

(110.5)

π-sandbox

$\frac{\pi}{2} =$

$$1+\frac{1}{3}*\left(1+\frac{2}{5}*\left(1+\frac{3}{7}*\left(1+\frac{4}{9}*\left(1+\frac{5}{11}*\left(1+\frac{6}{13}*(1+\cdots)\right)\right)\right)\right)\right) \qquad (120.1)$$

$$\left(\frac{\pi}{3}\right)^2 =$$

$$1+\frac{1}{3}*\frac{1}{4}*\left(1+\frac{2}{5}*\frac{2}{6}*\left(1+\frac{3}{7}*\frac{3}{8}*\left(1+\frac{4}{9}*\frac{4}{10}\left(1+\frac{5}{11}*\frac{5}{12}*\left(1+\frac{6}{13}*\frac{6}{14}*(1+\cdots)\right)\right)\right)\right)\right) \qquad (120.2)$$

What about an extension "$\frac{\pi^k}{?(k)}$ = infinite ne(k)“ ?

Lateral 1

A relationship between π and Φ

$$1-\frac{1}{3}*\left(1-\frac{2}{5}*\left(1-\frac{3}{7}*\left(1-\frac{4}{9}*\left(1-\frac{5}{11}*\left(1-\frac{6}{13}*(1-\cdots)\right)\right)\right)\right)\right) = 0{,}760862702\ldots = \sqrt{\frac{5}{2}}*\ln(\Phi) \qquad (121.1)$$

$$1-\frac{1}{3}*\frac{1}{4}*\left(1-\frac{2}{5}*\frac{2}{6}*\left(1-\frac{3}{7}*\frac{3}{8}*\left(1-\frac{4}{9}*\frac{4}{10}\left(1-\frac{5}{11}*\frac{5}{12}*\left(1-\frac{6}{13}*\frac{6}{14}*(1-\cdots)\right)\right)\right)\right)\right)=0.926259282\ldots$$

$$=4*(\ln(\Phi))^2 \qquad (121.2)$$

Lateral 2

What about other sign sequences, f.e.

$$1+\frac{1}{3}*\left(1-\frac{2}{5}*\left(1+\frac{3}{7}*\left(1-\frac{4}{9}*\left(1+\frac{5}{11}*\left(1-\frac{6}{13}*(1+\cdots)\right)\right)\right)\right)\right)=1.17339057028699270693328260541411219976\ldots$$

(122.1)

$$1-\frac{1}{3}*\left(1+\frac{2}{5}*\left(1-\frac{3}{7}*\left(1+\frac{4}{9}*\left(1-\frac{5}{11}*\left(1+\frac{6}{13}*(1-\cdots)\right)\right)\right)\right)\right)=0.60199105113553850399852545774212393040 52\ldots$$

(122.2)

or

$(++--)\text{per}$, $(+--++++-)\text{per}$, … ,

and sign sequences representing a binary message = encoding a binary plain text into equ. (122); find the related decoding algorithm, see

** **Unit 52, Small challenges to go, part 3, Ch 8**

** **Unit Lambda.1** ?

Lateral 3

What other mathematical constants [5,6,7] can be incorporated into the ine scheme equs (121,122)?

Sacra Conversazione

** Riemann Zeta function [3,4]

$$\zeta(2) = \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \qquad (123)$$

** Euler product [8]

$$\prod_{i=1}^{\infty} \frac{1}{1+\frac{1}{p_i^2}} = \pi^2 * \begin{cases} \frac{1}{6} \\ \frac{1}{15} \end{cases}$$

$$\rightarrow \prod_{i=1}^{\infty} \frac{p_i^2-1}{p_i^2+1} = \prod_{i=1}^{\infty} \frac{p_i-\frac{1}{p_i}}{p_i+\frac{1}{p_i}} = \frac{2}{5}$$

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11,\ \ldots \qquad (124)$$

More

$$\prod_{i=1}^{\infty} \frac{p_i^{2\lambda}}{p_i^{2\lambda}-1} = \zeta(2\lambda) = \frac{1}{2} * \frac{\pi^{2\lambda}}{2^{2\lambda}-1} * \delta(\lambda) \qquad (125.1)$$

$$\prod_{i=1}^{\infty} \frac{p_i^{2\lambda}}{p_i^{2\lambda}+1} = \frac{\pi^{2\lambda}}{2^{2\lambda}+1} * \frac{\delta(2\lambda)}{\delta(\lambda)} \quad [9] \qquad (125.2)$$

with

$$\delta(m=5) = \begin{vmatrix} \mathbf{1} & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{3} & \mathbf{2} & \mathbf{0} & 0 & 0 \\ \mathbf{5} & \mathbf{4} & \mathbf{2} & \mathbf{0} & 0 \\ \mathbf{7} & \mathbf{6} & \mathbf{4} & \mathbf{2} & \mathbf{0} \\ \mathbf{9} & \mathbf{8} & \mathbf{6} & \mathbf{4} & \mathbf{2} \end{vmatrix}; \qquad (125.3)$$

m*m determinant

$$\mathbf{k} = \frac{1}{k!}, \ \mathbf{0} = \frac{1}{0!} = 1, \ 0 = 0$$

**

$$\frac{\boldsymbol{\pi^2}}{12} = \cfrac{1}{1+\cfrac{1^4}{3+\cfrac{2^4}{5+\cfrac{3^4}{7+\cfrac{4^4}{9+\cfrac{5^4}{11+\cfrac{6^4}{13+\cdots}}}}}}} \qquad (126)$$

**

$$\frac{\boldsymbol{\pi^2}}{9} = 1 + \frac{1^2}{3*4} * \left(1 + \frac{2^2}{5*6} * \left(1 + \frac{3^2}{7*8} * \left(1 + \frac{4^2}{9*10} * \right. \right. \right.$$

$$\left. \left. \left. \left(1 + \frac{5^2}{11*12} * \left(1 + \frac{6^2}{13*14} * (1 + \cdots) \right) \right) \right) \right) \right) \qquad (127)$$

**

$$d(n=5) = \begin{vmatrix} \mathbf{2} & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{4} & \mathbf{2} & \mathbf{0} & 0 & 0 \\ \mathbf{6} & \mathbf{4} & \mathbf{2} & \mathbf{0} & 0 \\ \mathbf{8} & \mathbf{6} & \mathbf{4} & \mathbf{2} & \mathbf{0} \\ \mathbf{10} & \mathbf{8} & \mathbf{6} & \mathbf{4} & \mathbf{2} \end{vmatrix} ; \mathbf{k}=\frac{1}{k!} , \mathbf{0}=\frac{1}{0!}=1, 0=0$$

n*n determinant

$$\lim_{n\to\infty} \frac{d(n)}{d(n+1)} = \frac{\boldsymbol{\pi^2}}{4} \qquad (128)$$

** Pascal‘s triangle determinant and $\boldsymbol{\pi^2}$

$dPt(n=10) =$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \end{vmatrix} \rightarrow$$

n*n Pascal’s triangle determinant

↓↓ (129.1)

$$\text{dgPt}(n=7) = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 15 & 15 & 1 & 0 \\ 0 & 0 & 0 & 7 & 35 & 21 & 0 \\ 0 & 0 & 0 & 1 & 28 & 70 & 28 \end{vmatrix} \rightarrow$$

n*n reduced Pascal's triangle determinant

↓

(129.2)

$$f(n) = \binom{2n+4}{2} * \frac{\text{dgPt}(n)}{\text{dgPt}(n+1)} \qquad (129.3)$$

$$\lim_{n\to\infty} f(n) = \frac{\pi^2}{2} \qquad (129.4)$$

First difference quotient of f(n)

$$D1f(n) = \frac{f(n)-f(n+1)}{f(n+1)-f(n+2)} \qquad (129.5)$$

$$\lim_{n\to\infty} D1f(n) = D1f = 9 \qquad (129.6)$$

Linear convergence acceleration with D1f

$$f1l(n) = \frac{D1f}{D1f-1} * f(n+1) - \frac{1}{D1f-1} * f(n) \qquad (129.7)$$

Nonlinear convergence acceleration

$$f1nl(n) = \frac{\begin{vmatrix} f(n) & f(n+1) \\ f(n+1) & f(n+2) \end{vmatrix}}{f(n)+f(n+2)-2f(n+1)} \qquad (129.8)$$

Related execution record (10 digits)

n	f(n)	f1l(n)	f1nl(n)

1	5.0	4.933823529.	4.934873949
2	4.941176470	4.934772295	4.934804570
3	4.935483870	4.934801129	4.934802286
4	4.934876989	4.934793884	4.934802203
5	4.934803118	4.934802200	4.934802200
6	4.934802302	4.934802200	4.934802200
7	4.934802211		
8	4.934802201		
9	4.934802200		
10	4.934802200		

(129.9)

$$\text{dbp}(n=7) = \begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 20 & 6 & 0 & 0 \\ 0 & 0 & 1 & 21 & 35 & 7 & 0 \\ 0 & 0 & 0 & 8 & 56 & 56 & 8 \\ 0 & 0 & 0 & 1 & 36 & 126 & 84 \end{vmatrix}$$

n*n determinant, reduced Pascal's triangle

(130.1)

$$h(n) = \binom{2n+6}{3} * \frac{\text{dbPt}(n)}{\text{dbPt}(n+1)} \qquad (130.2)$$

$$\lim_{n\to\infty} h(n) = \frac{4}{3}\pi^2 \qquad (130.3)$$

First difference quotient

$$D1h(n) = \frac{h(n)-h(n+1)}{h(n+1)-h(n+2)} \qquad (130.4)$$

$$\lim_{n\to\infty} D1h(n) = D1h = 9$$

Linear convergence acceleration with D1f

$$h1l(n) = \frac{D1h}{D1h-1} * h(n+1) - \frac{1}{D1h-1} * h(n) \qquad (130.5)$$

Nonlinear convergence acceleration

$$h1nl(n) = \frac{\begin{vmatrix} h(n) & h(n+1) \\ h(n+1) & h(n+2) \end{vmatrix}}{h(n)+h(n+2)-2h(n+1)} \qquad (130.6)$$

Check the related execution records!

Open problems waiting for ACDG.net visitors

** Analyze the effect of the triple operator

$$f1(n) = \frac{\begin{vmatrix} f(n) & f(n+1) \\ f(n+1) & f(n+2) \end{vmatrix}}{f(n)+f(n+2)-2f(n+1)} \qquad (131)$$

on characteristic discrete functions f(n). Discuss extensions of this operator. (see differential calculus).

** In addition to π^2 also π, e, Φ, the Riemann Zeta function, and … are „hidden“ in Pascal’s triangle. Find out!
See also [10,11,12]

** **Wallis product** and surroundings

$$\frac{2*2}{1*3} * \frac{4*4}{3*5} * \frac{6*6}{5*7} * \frac{8*8}{7*9} * \frac{10*10}{9*11} * \frac{12*12}{11*13} * \frac{14*14}{13*15} *\ldots = \frac{\pi}{2}$$

(132)

1) $\frac{2*2}{1*3} * \frac{3*3}{2*4} * \frac{4*4}{3*5} * \frac{5*5}{4*6} * \frac{6*6}{5*7} * \frac{7*7}{6*8} * \frac{8*8}{7*9} *\ldots = 2$

2) $\frac{3*3}{2*4} * \frac{5*5}{4*6} * \frac{7*7}{6*8} * \frac{9*9}{8*10} * \frac{11*11}{10*12} * \frac{13*13}{12*14} * \frac{15*15}{14*16} *\ldots = \frac{4}{\pi}$

3) $\frac{1*3}{2*4} * \frac{5*7}{6*8} * \frac{9*11}{10*12} * \frac{13*15}{14*16} * \frac{17*19}{18*20} * \frac{21*23}{22*24} * \ldots = ?$

4) $\frac{1}{2} * \frac{4}{5} * \frac{9}{10} * \frac{16}{17} * \frac{25}{26} * \frac{36}{37} * \frac{49}{50} * \ldots = \frac{\pi}{\sinh(\pi)}$

5) $\frac{3}{5} * \frac{8}{10} * \frac{15}{17} * \frac{24}{26} * \frac{35}{37} * \frac{48}{50} * \frac{63}{65} \ldots = \frac{\pi}{\sinh(\pi)}$

6) $\frac{1}{2} * \frac{9}{10} * \frac{25}{26} * \frac{49}{50} * \frac{81}{82} * \frac{121}{122} * \frac{169}{170} \ldots = \frac{1}{\cosh(\frac{\pi}{2})}$

7) $\frac{4}{5} * \frac{16}{17} * \frac{36}{37} * \frac{64}{65} * \frac{100}{101} * \frac{144}{145} * \frac{196}{197} \ldots = \frac{\pi}{2\sinh(\frac{\pi}{2})}$

Extend this collection

Free running:

8) $\frac{1*5}{2*4} * \frac{9*13}{10*12} * \frac{17*21}{18*20} * \frac{25*29}{26*28} * \frac{33*37}{34*36} * \frac{41*45}{42*44} * \ldots =$
0.5946036132… = ?

9) $\frac{2*6}{3*5} * \frac{10*14}{11*13} * \frac{18*22}{19*21} * \frac{26*30}{27*29} * \frac{34*38}{35*37} * \frac{42*46}{43*45} * \ldots =$
0.7653669364… = ?

10) $\frac{2*6}{1*7} * \frac{10*14}{9*15} * \frac{18*22}{17*23} * \frac{26*30}{25*31} * \frac{34*38}{33*39} * \frac{42*46}{41*47} * \ldots =$

1.847758776… = ?

11) $\frac{7}{9} * \frac{26}{28} * \frac{63}{65} * \frac{124}{126} * \frac{215}{217} * \frac{342}{344} * \frac{511}{513} * \frac{728}{730} * \ldots = \frac{2}{3}$

12) $\prod_{k=2}^{\infty} \frac{p_k}{p_k + (-1)^{\frac{p_k - 1}{2}}} =$

$\frac{3}{2} * \frac{5}{6} * \frac{7}{6} * \frac{11}{10} * \frac{13}{14} * \frac{17}{18} * \frac{19}{18} * \frac{23}{22} * \ldots = \frac{\pi}{2}$

14) $\prod_{k=2}^{\infty} \frac{(2k+1)}{(2k+1)+(-1)^k} =$

$\frac{1*3}{2*2} * \frac{5*7}{6*6} * \frac{9*11}{10*10} * \frac{13*15}{14*14} * \frac{17*19}{18*18} * \frac{21*23}{22*22} * \ldots = \frac{1}{\sqrt{2}}$

(133.1-14)

15) Go to [14,15,16], Unit 193.4.3.Lateral Investigate convergence behaviour, difference quotient, and convergence acceleration of these products.

** Is it possible to construct a real function

$y = f(x);\ x,y \in \mathbb{R}_+$, (134)

represented by a formula expression or an algorithm,

with

$int(f(n)) = p_n$; $n \in \mathbb{N}$; $p_1=2, p_2=3, p_3=5, p_4=7, p_5=11, \ldots$?

Interlude 3

3.1

Transformation of the nested expression

$$A_0 + \frac{A_1}{B_1}(C_1 + \frac{A_2}{B_2}(C_2 + \frac{A_3}{B_3}(C_3 + \frac{A_4}{B_4}(C_4 + \ldots)))) =$$

(135.1)

into the continued fraction

$$cf = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{-a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{-a_4 + \cfrac{b_5}{a_5 + \cfrac{b_6}{-a_6 + \cdots}}}}}} \qquad (135.2)$$

with

$a_0 = A_0$

$a_1 = B_1$, $b_1 = A_1C_1$

$a_2 = A_2C_2 + B_2C_1$, $b_2 = A_2B_1C_2$

$a_i = A_iC_i + B_iC_{i-1}$, $b_i = A_iB_{i-1}C_{i-2}C_i$; i=3,4,5,…

(135.3)

3.2

Special case $C_1=C_2=C_3=C_4=\ldots=1$

$$A_0 + \frac{A_1}{B_1}\left(1 + \frac{A_2}{B_2}\left(1 + \frac{A_3}{B_3}\left(1 + \frac{A_4}{B_4}(1 + \ldots)\right)\right)\right) =$$

$$A_o + \cfrac{A_1}{+B_1 + \cfrac{A_2B_1}{-(A_2+B_2) + \cfrac{A_3B_2}{+(A_3+B_3) + \cfrac{A_4B_3}{-(A_4+B_4) + \cfrac{A_5B_4}{+(A_5+B_5) + + \cfrac{A_6B_5}{-(A_6+B_6)+\cdots}}}}}}$$

(136)

Example 12

$\sqrt{2} =$

$$1+\frac{2}{5}\left(1+\frac{1}{29}\left(1 + \frac{5}{169}\left(1 + \frac{29}{985}\left(1 + \frac{169}{5741+\cdots}(\ \ldots)\right)\right)\right)\right) =$$

$$1+\cfrac{2}{+5+\cfrac{1*5}{-(1+29)+\cfrac{5*29}{+(5+169)+\cfrac{29*169}{-(29+985)+\cfrac{169*985}{+(169+5741)+\cdots}}}}}$$

(137)

Example 14

$$\frac{\pi}{2}=$$

$$1+\frac{1}{3}*\left(1+\frac{2}{5}*\left(1+\frac{3}{7}*\left(1+\frac{4}{9}*\left(1+\frac{5}{11}*\left(1+\frac{6}{13}*\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.(1+\cdots)\right)\right)\right)\right)=$$

$$1+\cfrac{1}{+3+\cfrac{2*3}{-(2+5)+\cfrac{3*5}{+(3+7)+\cfrac{4*7}{-(4+9)+\cfrac{5*9}{+(5+11)++\cfrac{6*11}{-(6+13)+\cdots}}}}}}$$

(138)

Example 15

$$\left(\frac{\pi}{3}\right)^2=$$

$$1+\frac{1}{3}*\frac{1}{4}*\left(1+\frac{2}{5}*\frac{2}{6}*\left(1+\frac{3}{7}*\frac{3}{8}*\left(1+\frac{4}{9}*\frac{4}{10}\left(1+\frac{5}{11}*\frac{5}{12}*\left(1+\frac{6}{13}*\frac{6}{14}*(1+\cdots)\right)\right)\right)\right)\right)=$$

$$1+\cfrac{1}{3*4+\cfrac{2^2*3*4}{-(2^2+5*6)+\cfrac{3^2*5*6}{(3^2+7*8)+\cfrac{4^2*7*8}{-(4^2+9*10)+\cfrac{5^2*9*10}{(5^2+11*12)+\cfrac{6^2*11*12}{-(6^2+13*14)+\cdots}}}}}}=$$

$$1+\cfrac{1}{3*4+\cfrac{2*2^3*3}{-(3^2+5^2)+\cfrac{2*3^3*5}{(4^2+7^2)+\cfrac{2*4^3*7}{-(5^2+9^2)+\cfrac{2*5^3*9}{(6^2+11^2)+\cfrac{2*6^3*11}{-(7^2+13^2)+\cdots}}}}}}$$

(139)

3.3

Transformation of the power series

$$a_0+\frac{a_1}{b_1}x+\frac{a_2}{b_2}x^2+\frac{a_3}{b_3}x^3+\frac{a_4}{b_4}x^4+\frac{a_5}{b_5}x^5+\frac{6}{b_6}x^6+\ldots=$$

(140.1)

into the continued fraction

$$a_0+\cfrac{a_1x}{+b_1+\cfrac{a_2b_1^2x}{-(a_1b_2+a_2b_1x)+\cfrac{a_1a_3b_2^2x}{+(a_2b_3+a_3b_2x)+\cfrac{a_2a_4b_3^2x}{-(a_3b_4+a_4b_3x)+\cdots}}}}$$

(140.2)

Example 16

$$e^x = 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\ldots=$$

$$1+\cfrac{x}{+(1!)+\cfrac{(1!)^2x}{-(1!)(2+x)+\cfrac{(2!)^2x}{+(2!)(3+x)+\cfrac{(3!)^2x}{-(3!)(4+x)+\cfrac{(4!)^2x}{+(4!)(5+x)+\cdots}}}}}$$

(141)

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