Unit n°51

Multiple-argument formulas of the

$\left\{\begin{matrix}\text{trigonometric}\\ \text{hyperbolic}\end{matrix}\right.$ functions, esp. tangent,

in different representations. Find more.

51.1

Multiple-argument formulas of the

$\left\{\begin{matrix}\text{trigonometric}\\ \text{hyperbolic}\end{matrix}\right.$ tangent in different representations

51.1.1

2 by 2 matrix chain product representation

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & -(1+v*a)\\ 1 & 2\end{pmatrix}^{n\text{-}1}\begin{pmatrix}0\\ 1\end{pmatrix}=\begin{pmatrix}B(n)\\ A(n)\end{pmatrix};$$

$$v = \begin{cases}+1\\ -1\end{cases} \rightarrow$$

$$a=\begin{cases}\tan^2(x)\\ \tanh^2(x)\end{cases} \underline{\text{and}} \quad \frac{B(n)}{A(n)}=\begin{cases}\dfrac{\tan(nx)}{\tan x}\\ \dfrac{\tanh(nx)}{\tanh x}\end{cases};$$

$$n = 1,\ 2,\ 3,\ \ldots\ ;\quad \tanh x = \frac{\tan(ix)}{i} \tag{1}$$

51.1.2

Recurrence relations

$$\frac{\tan(nx)}{\tan x} = \frac{1+\dfrac{\tan[(n-1)x]}{\tan x}}{1-a\dfrac{\tan[(n-1)x]}{\tan x}};\qquad a=\tan^2 x \tag{2.1}$$

$$\frac{\tanh(nx)}{\tanh x} = \frac{1+\dfrac{\tanh[(n-1)x]}{\tanh x}}{1+a\dfrac{\tanh[(n-1)x]}{\tanh x}};\qquad a=\tanh^2 x \tag{2.2}$$

51.1.3

Representation by rational expressions

$$\frac{\tan(nx)}{\tan x} = \frac{\binom{n}{1}-\binom{n}{3}a+\binom{n}{5}a^2-\binom{n}{7}a^3+\binom{n}{9}a^4 \mp \ldots}{\binom{n}{0}-\binom{n}{2}a+\binom{n}{4}a^2-\binom{n}{6}a^3+\binom{n}{8}a^4 \mp \ldots};$$

$$a = \tan^2 x \; ; \; \binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!} & \text{for } \begin{matrix} n \geq m \\ n < m \end{matrix} \\ 0 \end{cases} \tag{3.1}$$

$$\frac{\tanh(nx)}{\tanh x} = \frac{\binom{n}{1} + \binom{n}{3} a + \binom{n}{5} a^2 + \binom{n}{7} a^3 + \binom{n}{9} a^4 + \ldots}{\binom{n}{0} + \binom{n}{2} a + \binom{n}{4} a^2 + \binom{n}{6} a^3 + \binom{n}{8} a^4 + \ldots} \; ;$$

$$a = \tanh^2 x \; ; \; \binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!} & \textit{for } \begin{matrix} n^3 m \\ n < m \end{matrix} \\ 0 \end{cases} \tag{3.2}$$

Representation by rational expressions 2

$$\begin{cases} \dfrac{\tan(nx)}{\tan x} \\ \dfrac{\tanh(nx)}{\tanh x} \end{cases} = \frac{1}{\tau} \frac{(1+\tau)^n - (1-\tau)^n}{(1+\tau)^n + (1-\tau)^n} \text{ with } \begin{cases} \tau = i \tan x \\ \tau = \tanh x \end{cases} \tag{4}$$

51.1.4

Representation by determinants

$$A(n)=\begin{vmatrix} 1 & b & 0 & 0 & 0 & 0 & \rightarrow \\ 1 & 2 & b & 0 & 0 & 0 & \\ 0 & 1 & 2 & b & 0 & 0 & \\ 0 & 0 & 1 & 2 & b & 0 & \\ 0 & 0 & 0 & 1 & 2 & b & \\ 0 & 0 & 0 & 0 & 1 & 2 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

n by n determinant

$$B(n)=\begin{vmatrix} 1 & b & 0 & 0 & 0 & \rightarrow \\ 1 & 2 & b & 0 & 0 & \\ 0 & 1 & 2 & b & 0 & \\ 0 & 0 & 1 & 2 & b & \\ 0 & 0 & 0 & 1 & 2 & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

$(n-1)$ by $(n-1)$ determinant

$$b = 1 \pm a = \begin{cases} 1+\tan^2 x \\ 1-\tanh^2 x \end{cases} \rightarrow \frac{B(n)}{A(n)} = \begin{cases} \dfrac{\tan(nx)}{\tan x} \\ \dfrac{\tanh(nx)}{\tanh x} \end{cases} \qquad (5)$$

Representation by infinite power series in determinant notation

$$\begin{Bmatrix} \dfrac{\tan(nx)}{\tan x} \\ \dfrac{\tanh(nx)}{\tanh x} \end{Bmatrix} =$$

$$-\begin{vmatrix}
0 & \binom{n}{1} & \mp\binom{n}{3} & \binom{n}{5} & \mp\binom{n}{7} & \binom{n}{9} & \mp\binom{n}{11} & \rightarrow \\
1 & \binom{n}{0} & \mp\binom{n}{2} & \binom{n}{4} & \mp\binom{n}{6} & \binom{n}{8} & \mp\binom{n}{10} & \\
a & 0 & \binom{n}{0} & \mp\binom{n}{2} & \binom{n}{4} & \mp\binom{n}{6} & \binom{n}{8} & \\
a^2 & 0 & 0 & \binom{n}{0} & \mp\binom{n}{2} & \binom{n}{4} & \mp\binom{n}{6} & \\
a^3 & 0 & 0 & 0 & \binom{n}{0} & \mp\binom{n}{2} & \binom{n}{4} & \\
a^4 & 0 & 0 & 0 & 0 & \binom{n}{0} & \mp\binom{n}{2} & \\
a^5 & 0 & 0 & 0 & 0 & 0 & \binom{n}{0} & \\
\downarrow & & & & & & & \searrow
\end{vmatrix};$$

$$\pm \rightarrow a = \begin{Bmatrix} \tan^2 x \\ \tanh^2 x \end{Bmatrix}; \quad \binom{n}{m} = \begin{Bmatrix} \dfrac{n!}{m!(n-m)!} & \text{for} & n \geq m \\ 0 & & n < m \end{Bmatrix} \qquad (6)$$

Example n=3

$$\frac{\tanh(3x)}{\tanh x} = -\begin{vmatrix} 0 & 3 & 1 & 0 & 0 & \rightarrow \\ 1 & 1 & 3 & 0 & 0 & \\ a & 0 & 1 & 3 & 0 & \\ a^2 & 0 & 0 & 1 & 3 & \\ a^3 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

$$= 3 - 8a + 24a^2 - 72a^3 + 216a^4 \mp \ldots$$

$$= 3 - 8a[1 - (3a) + (3a)^2 - (3a)^3 \pm \ldots]$$

$$x = 0.2 \rightarrow \quad a = \tanh^2 x \quad = 0.038957017\ldots$$

$$\frac{\tanh(3x)}{\tanh x} \quad = 2.720956026\ldots \qquad (7)$$

$$3 - 8a[1 - (3a) + (3a)^2 - (3a)^3] \quad = 2.721015183\ldots$$

51.2

Multiple-argument formulas of the

$$\begin{cases} \text{trigonometric} \\ \text{hyperbolic} \end{cases}$$ sine and cosine in different representations:

$\sin(nx)$, $\cos(nx)$, $\sinh(nx)$, $\cosh(nx)$ =

It is your turn, my student.

And more:

$$\tan\left(n\tan^{-1}x\right),\ \tanh\left(n\tanh^{-1}x\right),$$
$$\sin\left(n\sin^{-1}x\right),\ \sinh(n\sinh^{-1}x),$$
$$\cos\left(n\cos^{-1}x\right),\ \cosh\left(n\cosh^{-1}x\right),\ \ldots =$$

Hint

$$\cos(n * \arccos x) \;=\; \cos(n * \cos^{-1} x) \;=\; T_n(x)$$

Chebyshev polynomial of the first kind

$$\begin{pmatrix} T_{n+1} \\ T_n \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix} \tag{8.1}$$

$$= \begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} T_1 \\ T_0 \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$T_n = \begin{vmatrix} x & 1 & 0 & 0 & 0 & \rightarrow \\ 1 & 2x & 1 & 0 & 0 & \\ 0 & 1 & 2x & 1 & 0 & \\ 0 & 0 & 1 & 2x & 1 & \\ 0 & 0 & 0 & 1 & 2x & \\ \downarrow & & & & & \searrow \end{vmatrix} \tag{8.2}$$

n by *n* determinant

References

[1] http://mathworld.wolfram.com/Multiple-AngleFormulas.html

[2] http://en.wikipedia.org/wiki/List_of_trigonometric_identities#Multiple-angle_formulae

[3] http://mathworld.wolfram.com/Tangent.html

[4] http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html