# Unit n°5

## Small creative algorithmic finger exercises

Small creative algorithmic finger exercises in the style of J.S. Bach's Well-Tempered Clavier (composed "for the profit and use of musical youth desirous of learning, and especially for the pastime of those already skilled in this study")

Algorithm = well-defined structured procedure

- to solve a problem class in a finite number of steps, but also
- to describe a class of time-discrete terminating processes.

### Examples

### 5.1 Eadem mutata resurgo

Find a simple and fast algorithm for the numeric calculation of

$$x = \sqrt{2} + \sqrt{3} = 3.1462643\ldots . \qquad (1)$$

Solution

$$x = \sqrt{2} + \sqrt{3} \rightarrow \quad x^2 = 2 + 2\sqrt{6} + 3 \rightarrow \left(x^2 - 5\right)^2 = 24 \rightarrow$$

$$x^4 - 10x^2 + 1 = 0 \rightarrow x = \frac{(10x^2 - 1)}{x^3}$$

$\rightarrow$ "Babylon" approach:

$$x := \frac{1}{2}\left(x + \frac{10x^2 - 1}{x^3}\right) \rightarrow \qquad (2)$$

| $t$ | $x(t)$ |
|---|---|
| 0 | 5.0 |
| 1 | 3.496 |
| 2 | 3.166504067 |
| 3 | 3.146532320 |
| 4 | 3.146267115 |
| 5 | 3.146264398 |
| 6 | 3.146264370 |
| 7 | 3.146264370 |

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)} = \lim_{t\to\infty}\left(x(t)\right)^4 = 97.98979485566.$$

$$= \left(\sqrt{2}+\sqrt{3}\right)^4 \qquad (3.1)$$

Check $\lim_{t\to\infty} d^{(k)}x(t)$

$$d^{(k)}x(t) = \frac{d^{(k-1)}x(t)-d^{(k-1)}x(t+1)}{d^{(k-1)}x(t+1)-d^{(k-1)}x(t+2)}$$

$$k = 2,3,4,5,\ldots \qquad (3.2)$$

$$d^{(0)}x(t) = x(t)$$

$$d^{(1)}x(t) = dx(t)$$

Diskussion!

An example for a converging recursion equation

$$x := \varphi(x) \qquad (4)$$

with

$$\lim_{t\to\infty} x(t) = a \qquad (5)$$

and

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)} = a^n; n\in\mathbb{N} \qquad (6)$$

** Find more examples for the function $\varphi(x)$ that fulfils in the recursion equation (4) the conditions (5) and(6).
** Find a general solution for this problem.
** Are there possible applications in science and engineering?

## 5.2 Tertium datur

$$a(x,y)=\frac{x+y}{2}\geq h(x,y)=\frac{2xy}{x+y}\geq g(x,y)=\sqrt{xy}\ \ ;\ \ x,y\in\mathbb{R}_+$$

That means, the value of the harmonic mean is between arithmetic and geometric mean.

Is it possible to insert in a similar way between addition and multiplication a third commutative and associative arithmetic operation $a\square b$ with

$a+b\leq a\square b\leq a\cdot b\ ;\ a,b\in\{2,3,4,\ldots\}$ ?

Are there possible applications for this operation $a\square b$ in science and engineering?

## 5.3 Design a partition algorithm for the following problem.

Given a natural number n.
Find two natural numbers *k* and *m* with

** $k\cdot m=n$
** $k\geq m$
** the smallest possible difference $k-m$

Example

$n=1234567890,\ k=36070,\ m=34227$

Special cases

** $k=m\ \rightarrow$ square root extraction $k=m=\sqrt{n}$
** $n=a^2-b^2;\ a\gg b\rightarrow k=a+b;\ m=a-b$

** $n$ is semiprim, e.g. $k$ and $m$ are prime → RSA code breaking (semiprime factorization)
** $n$ is prime, e.g. $k = n$ , $m = 1$ → primality test
** ?

## 5.4 Selbstkorrektur?

Wir betrachten die “innere Struktur“ der Dezimaldarstellung transzendenter mathematischer Konstanten

$e = 2.718281828\ldots$
$\pi = 3.141592654\ldots$

etc.

Ist eine Verfälschung

* einer Digitalstelle (Einfachfehler)
* von $k$ aufeinanderfolgenden Digitalstellen (Bündelfehler) in beliebiger Position mittels eines Algorithmus, der allein die der Fehlerstelle\n benachbarten Dezimalstellen als Input einbezieht,

    ** erkennbar
    ** korrigierbar?

Mit anderen Worten:
enthalten diese Konstanten in Dezimaldarstellung einen verteilten Selbstkorrekturmechanismus (vgl. fehlerkorrigierende Kodes in Informatik und Nachrichtentechnik; Selbstreparatur im genetischen Kode)?

## 5.5

We know the following formula pairs:

| Compact expression | long-tailed sum expression | with the coefficient display |
|---|---|---|
| 1<br>$(x+y)^n =$ | $\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ | 1<br>1 1<br>1 2 1<br>1 3 3 1<br>1 4 4 4 1<br>⋮ |
| 2<br>$\frac{x^{n+1}-y^{n+1}}{x-y} =$ | $\sum_{i=0}^{n} x^i y^{n-i}$ | 1<br>1 1<br>1 1 1<br>1 1 1 1<br>1 1 1 1 1<br>⋮ |
| 3<br>$(x+y)\frac{x^n-y^n}{x-y} =$ | $x^n + 2\sum_{i=1}^{n-1} x^i y^{n-i} + y^n$ | 1 1<br>1 2 1<br>1 2 2 1<br>1 2 2 1 1<br>⋮ |
| 4<br>$\frac{x^{m+1}-y^{m+1}}{x-y} \cdot$<br>$\frac{x^{n+1}-y^{n+1}}{x-y} =$ | $\sum_{i=0}^{m+n} (\min(i,m,n)+1) x^i y^{m+n-i}$ | m=2<br>n=0<br>1 1 1<br>1 2 2 1<br>2 1 2 3 2 1<br>3 1 2 3 3 2 1<br>4 1 2 3 3 3 2 1<br>⋮ |

| | | |
|---|---|---|
| 5<br>$\frac{x^{n+1}-(-y)^{n+1}}{x+y}=$ | $\sum_{i=0}^{n}(-1)^i x^i y^{n-i}$ | 1<br>1 -1<br>1 -1 1<br>1 -1 1 -1<br>1 -1 1 -1 1<br>⋮ |

** Find more, using the proof method “Mathematical induction” as heuristic tool.

** Discuss applications.