# Unit n°37

## Triangle Centers

We consider a planar triangle $\Delta P_1P_2P_3$

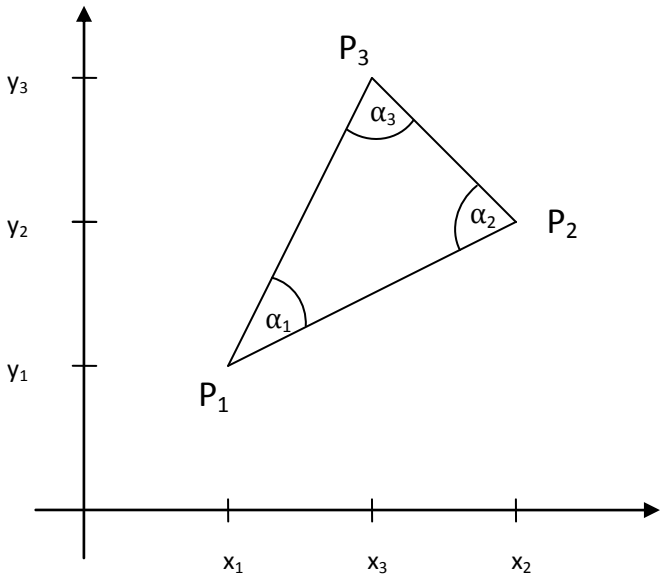



Fig. 1 *Planar triangle*

With the vertices $P_i = (x_i, y_i);\ i = 1,2,3$

and the angles $\alpha_1$, $\alpha_2$ and $\alpha_3$.

For this angles in function of the coordinates of the vertices we have in shortened determinant representation (Sarrus evaluation) [1]

$$\tan\alpha_1 = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 y_1 & x_2 y_2 \\ \cdots y_3 & x_2 y_2 & x_3 \cdots \end{vmatrix}} \qquad (1.1)$$

tan $\alpha_2$ and tan $\alpha_3$ can be obtained from equ.(1.1) by cyclic index permutation

$$1, 2, 3 \rightarrow 2, 3, 1 \rightarrow 3, 2, 1 \qquad (1.2, 1.3)$$

The coordinates of the classical centers $P = (x, y)$ of the triangle $\Delta\, P_1P_2P_3$ [2, 3, 4, 5]

* C centroid (geometric center)
  intersection of the three medians [6, 7]
* I incenter (center of the incircle)
  intersection of the three interior angle bisectors [8, 9]
* CC circumcenter (center of the circumscribed circle)

intersection of the three perpendicular bisectors of the sides [10, 11]

* O orthocenter

intersection of the three altitudes [12, 13]

are linear functions of the coordinates of the vertices $P_i = (x_i, y_i);\ i = 1,2,3$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} ; \quad w_1 + w_2 + w_3 = 1 \qquad (2.1)$$

inverse

$$w_1 = \frac{\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} ; \quad w_2 = \frac{\begin{vmatrix} x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} ; \quad w_3 = \frac{\begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad (2.2)$$

The weights $w_i$ for C, I, CC and O are normalized functions of the $\alpha_i$ :

With

$$(w_1, w_2, w_3) = w(f, n) = \left( \frac{f(n\alpha_1)}{\sum_{i=1}^{3} f(n\alpha_i)}, \frac{f(n\alpha_2)}{\sum_{i=1}^{3} f(n\alpha_i)}, \frac{f(n\alpha_3)}{\sum_{i=1}^{3} f(n\alpha_i)} \right) \qquad (3)$$

We have

| | | |
|---|---|---|
| P | $(w_1, w_2, w_3)$ | |
| C | $w(\cos, O) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ | (4.1) |
| I | $w(\sin, 1)$ | (4.3) |
| CC | $w(\sin, 2)$ | (4.3) |
| O | $w(\tan, 1)$ | (4.4) |

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**Problem**

1. Derive the given equs (1), (2), (3) and (4).
2. What triangle center is described by

| | | |
|---|---|---|
| 2.1 | $w(\sin, 3)$ | (5.1) |
| 2.2 | $w(\sin, \frac{1}{2})$ | (5.2) |
| 2.3 | $w(\sin^2, 1)$ | (5.3) |
| 2.4 | $w(\sinh, 1)$ | (5.4) |
| 2.5 | $w(\cos, 1)$ | (5.5) |
| 2.6 | $w(\tan, 2)$ | (5.6) |
| 2.7 | $w(\tan, \frac{1}{2})$ | (5.7) |

3. Find w(f, n) for the point K in fig.2 .

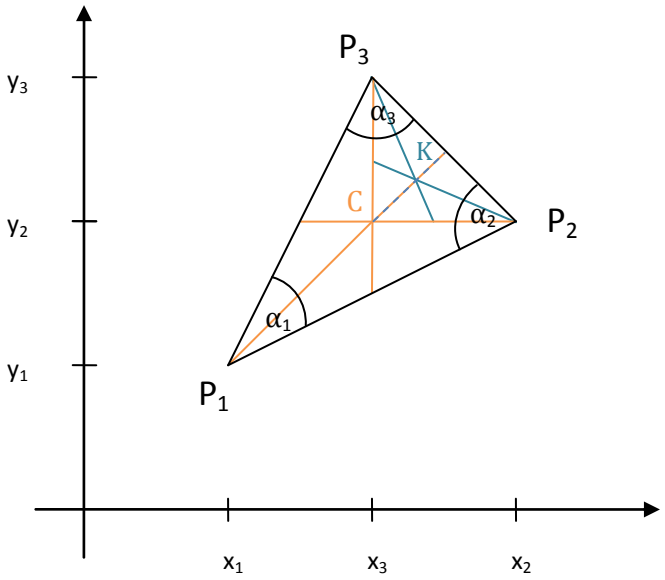



Fig. 2 *The centers C and K*
*all internal lines are medians*

## References


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[13] http://en.wikipedia.org/wiki/Orthocenter