Unit 2, example 6
Formula manipulation is an essential part of algorithm design
Classical and fast multiplication of natural numbers by hand

We all learnt in the primary school the classical multiplication procedure of natural numbers by hand.
Example 1

9 5 6 3 7 3 1 4 6 5 8 9 x 2 7 1 4 6 6 9 2	
1 1 1 0 1 0 0 0 1 1 1 1	carry
1 9 1 2 7 4 6 2 9 3 1 7 8	**digit**
6 3 4 2 5 2 1 3 4 4 6 6	carry
6 6 9 4 6 1 2 0 2 6 1 2 3	**digit**
0 0 0 0 0 0 0 0 0 0 0 0	.
9 5 6 3 7 3 1 4 6 5 8 9	**.**
3 2 2 1 2 1 0 1 2 2 3 3	
3 8 2 5 4 9 2 5 8 6 3 5 6	
5 3 3 2 4 1 0 2 3 3 5 5	
5 7 3 8 2 3 8 8 7 9 5 3 4	
5 3 3 2 4 1 0 2 3 3 5 5	
5 7 3 8 2 3 8 8 7 9 5 3 4	
8 5 5 3 6 2 1 4 5 5 8 8	.
8 6 0 7 3 5 8 3 1 9 3 0 1	**.**
1 1 1 0 1 0 0 0 1 1 1	carry
1 9 1 2 7 4 6 2 9 3 1 7 8	**digit**
1 0 2 3 3 3 3 4 2 3 5 5 4 3 3 1 0 0 0	carry
2 5 9 6 2 3 6 7 2 4 7 5 2 2 4 3 3 5 8 8	**digit**

Example 2
Multiplication of two natural numbers represented in a position system with the basis B, classical sequential procedure (carries not taking in consideration):

$a = a_3B^3+a_2B^2+a_1B^1+a_0B^0$; a,a_i,Bə.˙

$b = b_2B^2+b_1B^1+b_0B^0$; b,b_iə.˙ ; a←b

$a*b = c = c_5B^5+c_4B^4+c_3B^3+c_2B^2+c_1B^1+c_0B^0$; c,c_iə.˙

$a*b = (a_3B^3+a_2B^2+a_1B^1+a_0B^0)*(b_2B^2+b_1B^1+b_0B^0)$

$a_3b_2B^5+a_2b_2B^4+a_1b_2B^3+a_0b_2B^2$

$a_3b_1B^4+a_2b_1B^3+a_1b_1B^2+a_0b_1B^1$

$a_3b_0B^3+a_2b_0B^2+a_1b_0B^1+a_0b_0B^0$

$$a_3b_2B^5+(a_2b_2+a_3b_1)B^4+(a_1b_2+a_2b_1+a_3b_0)B^3+(a_0b_2+a_1b_1+a_2b_0)B^2+(a_0b_1+a_1b_0)B^1+a_0b_0B^0$$
$$= c_5\,B^5+ c_4\,B^4+ c_3\,B^3+ c_2\,B^2+ c_1\,B^1+ c_0\,B^0$$

The last two lines also can be interpreted by the following sequential calculation schedule (calculation graph) a*b → c (carries not taking in consideration):

0 0 0 a_3 a_2 a_1 a_0 0 0
---------- local operator in start position = step 0
go to next left □ | * + * + * | = $a_0*b_0+0*b_1+0*b_2 = a_0b_0 = c_0$
----
b_0 b_1 b_2 (reverse notation!)

0 0 0 a_3 a_2 a_1 a_0 0 0
---------- local operator in step 1
| * + * + * | = $a_1*b_0+a_0*b_1+0*b_2 = a_1b_0+a_0b_1 = c_1$
----
b_0 b_1 b_2

0 0 0 a_3 a_2 a_1 a_0 0 0 step 2
| * + * + * | = $a_2b_0+a_1b_1+a_0b_2 = c_2$
b_0 b_1 b_2

0 0 0 a_3 a_2 a_1 a_0 0 0 step 3
| * + * + * | = $a_3b_0+a_2b_1+a_1b_2 = c_3$
b_0 b_1 b_2

0 0 0 a_3 a_2 a_1 a_0 0 0 step 4
| * + * + * | = $a_3b_1+a_2b_2 = c_4$
b_0 b_1 b_2

0 0 0 a_3 a_2 a_1 a_0 0 0 step 5
| * + * + * | = $a_3b_2 = c_5$
b_0 b_1 b_2

0 0 0 a_3 a_2 a_1 a_0 0 0 step 6 = stop
* + * + * | = 0 → stop
b_0 b_1 b_2

This calculation schedule is the basis for a fast multiplication procedure (esp. for mental calculation, Trachtenberg [1]), demonstrated step by step by the following example.

Example 3
a = 7398 ; b = 389 ; B = 10 ; a*b = c = 2877822
Calculation schedule, carries taking in consideration

0 0 0 0 7 3 9 8. 0 0 x 3 8 9
---
0 0 0 0 7 3 9 8. 0 0

| * + * + * | = **8*9+0*8+0*3** = 7**2**

9 8 3 (reverse notation)

7 **2** carry **digit**

| * + * + * | = **9*9+8*8+0*3** + 7 = 15**2**

9 8 3

15 **2**

| * + * + * | = **3*9+9*8+8*3** + 15 = 13**8**

9 8 3

13 **8**

| * + * + * **|** = **7*9+3+8+9*3** + 13 = 12**7**

9 8 3

12 **7**

| * + * + * **|** = **9*0+7*8+3*3** + 12 = 7**7**

9 8 3

7 7

| * + * + * **|** = **0*9+0*8+7*3** + 7 = 2**8**

9 8 3

2 **8**

| * + * + * **|** = **0*0+0*0+0*0** + 2 = **2**

9 8 3

0 **2**

* + * + * **|** = **0*0+0*0+0*0** + 0 = **0** → stop

9 8 3

---

2 8 7 7 8 2 2

Condensed version:

0 0 0 0 7 3 9 8 # # ° 9 8 3 (reverse notation!)

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0 2 **7** 12 13 15 7

0 2 8 7 7 8 2 2

↓

stop

Input of the local operator in a given position (step) k:
- all m digits of factor b (in reverse notation),
- m homologous digits in factor a (including 0s),
- carry of the previous position (step) k-1.

The calculation of digit k and of the carry of step k takes place independently of the output digits k-1, k-2, .., 2, 1, 0

Output in position k:
- digit k,
- carry of step k to step k+1,
- shift instruction for the local operator: go left to position k+1.

Local operator: deterministic abstract automaton, basis for mental calculation.

Exercises

5.1

Integration of check methods for arithmetic calculations (casting out nines/elevens [3,4]) into the fast multiplication procedure.

5.2

Turing machine and structogram notation of the fast multiplication prodedure.

5.3

Extension: serial-parallel fast multiplication of 3 and more natural numbers by a multiprocessor system.

5.4

Fast division using a similar local operator?

References

[1] J.Trachtenberg. The Trachtenberg Speed System of Basic Mathematics. London 1994. ISBN 0285629166

[2] B.Handley. Speed Mathematics: Secret Skills for Quick Calculations. Wiley 2003. ISBN 0471467316

[3] Neuner- und Elferprobe. http://de.wikipedia.org/wiki/Neunerprobe

[4] Casting Out Nines. http://mathworld.wolfram.com/CastingOutNines.html