Example 4
Differential calculus

From the basic DC formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (1)$$

to the DC product rule

$$\left(f_1(x)*f_2(x)\right)' = f_1(x)*f_2'(x) + f_1'(x)*f_2(x) \qquad (2)$$

and to the DC product&quotient rule

$$\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)' = \frac{1}{\left(g(x)\right)^2} \begin{vmatrix} f_1'(x) & g'(x) & f_2'(x) \\ f_1(x) & g(x) & 0 \\ 0 & g(x) & f_2(x) \end{vmatrix} \qquad (3)$$

Equ. (1) gives for

$f(x) = f_1(x)*f_2(x)$

(4.1)

and insertion of the "seed" 0 into the numerator

$$\left(f_1(x)*f_2(x)\right)' = \lim_{h \to 0} \frac{f_1(x+h)*f_2(x+h) - f_1(x)*f_2(x) + 0}{h} \qquad (4.2)$$

Now we extend the seed 0 by an opposite pair

$0 = f_1(x+h)*f_2(x) - f_1(x+h)*f_2(x)$ [1)]

(4.3)

and obtain so from equ. (5.2) after rearrangement

$$\left(f_1(x)*f_2(x)\right)' =$$

$$\lim_{h\to 0}\frac{f_1(x+h)*f_2(x+h) - f_1(x+h)*f_2(x) + f_1(x+h)*f_2(x) - f_1(x)*f_2(x)}{h}$$

$$= \lim_{h\to 0}\frac{f_1(x+h)*f_2(x+h) - f_1(x+h)*f_2(x)}{h}$$

$$+ \lim_{h\to 0}\frac{f_1(x+h)*f_2(x) - f_1(x)*f_2(x)}{h}$$

$$= \lim_{h\to 0}\frac{f_1(x+h)*(f_2(x+h) - f_2(x))}{h} + \lim_{h\to 0}\frac{(f_1(x+h) - f_1(x))*f_2(x)}{h}$$

$$= f_1(x)*f_2'(x) + f_1'(x)*f_2(x)\ , \qquad (4.4)$$

the DC product rule.

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1) Equivalent alternative

$$0 = f_1(x)*f_2(x+h) - f_1(x)*f_2(x+h) \qquad (4.5)$$

Check it.

Now the same derivation for equ. (3)

$$\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{\frac{f_1(x+h)*f_2(x+h)}{g(x+h)} - \frac{f_1(x)*f_2(x)}{g(x)} + 0}{h}$$

$$= \lim_{h\to 0}\frac{f_1(x+h)f_2(x+h)g(x) - f_1(x)f_2(x)g(x+h) + 0}{g(x+h)g(x)h} \qquad (5.1)$$

Targeted extension of the seed 0 by opposite pairs

$$0 = f_1(x)f_2(x+h)g(x) - f_1(x)f_2(x+h)g(x)$$

$+ f_1(x)f_2(x)g(x) - f_1(x)f_2(x)g(x)$.

(5.2)

These equations provide after rearrangement

$$\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)' = \lim_{h\to 0}\frac{1}{g(x+h)g(x)h}\{[f_1(x+h)-f_1(x)]f_2(x+h)g(x)$$
$$+ f_1(x)[f_2(x+h)-f_2(x)]g(x)$$
$$- f_1(x)f_2(x)[g(x+h)-g(x)]\}$$
$$= \lim_{h\to 0}\frac{[f_1(x+h)-f_1(x)]f_2(x+h)g(x)}{g(x+h)g(x)h}$$
$$+ \lim_{h\to 0}\frac{f_1(x)[f_2(x+h)-f_2(x)]g(x)}{g(x+h)g(x)h}$$
$$- \lim_{h\to 0}\frac{f_1(x)f_2(x)[g(x+h)-g(x)]}{g(x+h)g(x)h}$$
$$= \frac{f_1'(x)f_2(x)g(x) + f_1(x)f_2'(x)g(x) - f_1(x)f_2(x)g'(x)}{[g(x)]^2}$$

$$= \frac{1}{[g(x)]^2}\begin{vmatrix} f_1'(x) & g'(x) & f_2'(x) \\ f_1(x) & g(x) & 0 \\ 0 & g(x) & f_2(x) \end{vmatrix} , \qquad (5.3)$$

the DC product&quotienten rule.

Find an algorithmus for an automatic determination of the “targeted extension of the seed 0 by opposite pairs” for a given

pro&quo expression like

$$\left(f_1(x)*f_2(x)\right)' \ , \ \left(\frac{f_1(x)*f_2(x)}{g(x)}\right)' \ , \ \left(\frac{f_1(x)*f_2(x)}{g_1(x)*g_2(x)}\right)' \ . \qquad (6)$$