Unit n° 2

Creative formula manipulation

Example 1 - Power sums and Pascal's triangle

Creative formula manipulation is based in a large part on mastering the dialectic between freedom and determination in arithmetic:

- Freedom:
  * Commutative and associative laws of addition and multiplication are the basis for a free arrangement (and re-arrangement) and free structured bracketing of operands.
  * Free and creative use of the two directions of the identity laws and its derivatives

$$a+0=a$$
$$a-a=0$$
$$a*1=a$$
$$a/a=1$$
$$a*0=0$$
$$(-1)*(-1)=1$$

← ---------- →

extension ← insertion reduction

- Determination:
  * Distributive law of multiplication with respect to addition
  * Priority law + < * < ^ .

Example 1
Power sums and Pascal's triangle

Now is $\sum_{i=1}^{n} i^4 \to \sum_{i=1}^{n} i^m = p_m(n)$; $m, n \in \mathbb{N}$ on the stage.

We start light-footed with a simple identity

$4^k = 4^k$; $k \in \mathbb{N}_0$

Now we insert the additive identity 0 (invariant)

$4^k = 4^k + 0 + 0 + 0 - 0$.

Next step: extension of 0 by an opposite pair

$4^k = 4^k + (3^k\text{-}3^k) + (2^k\text{-}2^k) + (1^k\text{-}1^k) - 0^k$:

* “creatio ex nihilo”

* Pilgering process.

* A chess-player plans always some moves in advance.

Last step: reordering:

$$4^k = (4^k\text{-}3^k) + (3^k\text{-}2^k) + (2^k\text{-}1^k) + (1^k\text{-}0^k) = \sum_{i=1}^{4} i^k - (i\text{-}1)^k \ .$$

Now generalization $4 \rightarrow k$:

$n^k = \sum_{i=1}^{n} i^k - (i-1)^k \ ; \ n \in \mathbb{N} \ ; k \in \mathbb{N}_0$

This is our magic formula.

Using the binomial formula, we obtain

$$n^k = \sum_{i=1}^{n} [i^k - (i-1)^k] = \sum_{i=1}^{n} [i^k - \sum_{j=0}^{k-1} (\binom{k}{j} (-1)^{k-j} i^j) - i^k] =$$

$$= \sum_{i=1}^{n} [\sum_{j=0}^{k-1} (\binom{k}{j} (-1)^{k-1-j} i^j)] = \sum_{j=0}^{k-1} (\binom{k}{j} (-1)^{k-1-j} [\sum_{i=1}^{n} i^j])$$

and with

$$\sum_{i=1}^{n} i^j = f(j,n)$$

$$n^k = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-1-j} f(j,n) \ ; \ k=1,2,3,4,5,.. \ ,$$

detailed for k=1(1)5

$$+n^1 = \binom{1}{0} f(0,n)$$
$$-n^2 = \binom{2}{0} f(0,n) - \binom{2}{1} f(1,n)$$
$$+n^3 = \binom{3}{0} f(0,n) - \binom{3}{1} f(1,n) + \binom{3}{2} f(2,n)$$
$$-n^4 = \binom{4}{0} f(0,n) - \binom{4}{1} f(1,n) + \binom{4}{2} f(2,n) - \binom{4}{3} f(3,n)$$
$$+n^5 = \binom{5}{0} f(0,n) - \binom{5}{1} f(1,n) + \binom{5}{2} f(2,n) - \binom{5}{3} f(3,n) + \binom{5}{4} f(4,n) \quad ,$$

in matrix representation

$$\begin{pmatrix} +n^1 \\ -n^2 \\ +n^3 \\ -n^4 \\ +n^5 \end{pmatrix} = \begin{pmatrix} +\binom{1}{0} & 0 & 0 & 0 & 0 \\ +\binom{2}{0} & -\binom{2}{1} & 0 & 0 & 0 \\ +\binom{3}{0} & -\binom{3}{1} & +\binom{3}{2} & 0 & 0 \\ +\binom{4}{0} & -\binom{4}{1} & +\binom{4}{2} & -\binom{4}{3} & 0 \\ +\binom{5}{0} & -\binom{5}{1} & +\binom{5}{2} & -\binom{5}{3} & +\binom{5}{4} \end{pmatrix} \begin{pmatrix} f(0,n) \\ f(1,n) \\ f(2,n) \\ f(3,n) \\ f(4,n) \end{pmatrix} .$$

Then f(4,n) can be found using Cramer's rule
http://mathworld.wolfram.com/CramersRule.html
after slight rearrangement as

$$f(4,n) = \sum_{i=1}^{n} i^4 = \frac{n}{(4+1)!} \begin{vmatrix} +n^0 & \binom{1}{0} & 0 & 0 & 0 \\ -n^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +n^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -n^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +n^{3+1} & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}$$

determinant of (4+1)th order

$$= \frac{n}{5!} \begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}$$

Pascal's triangle

determinant of 5th order

and after normalization

$$\frac{f(4,n)}{f(0,n)} = \frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0 = n} = \frac{\begin{vmatrix} +n^0 & \binom{1}{0} & 0 & 0 & 0 \\ -n^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +n^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -n^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +n^4 & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}}{\begin{vmatrix} +1 & \binom{1}{0} & 0 & 0 & 0 \\ -1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +1 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -1 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +1 & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}} = (4+1)!$$

determinants of (4+1)th order

$$= \frac{1}{5!}\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix} = \frac{\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} +1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ +1 & 1 & 3 & 3 & 0 \\ -1 & 1 & 4 & 6 & 4 \\ +1 & 1 & 5 & 10 & 10 \end{vmatrix}} \qquad \text{(c)}$$

determinants of 5th order

$$= \sum_{j=0}^{4} A(4,j)\, n^j = p_4(n)$$

$$= \begin{vmatrix} A(4,0) & -1 & 0 & 0 & 0 \\ A(4,1) & n & -1 & 0 & 0 \\ A(4,2) & 0 & n & -1 & 0 \\ A(4,3) & 0 & 0 & n & -1 \\ A(4,4) & 0 & 0 & 0 & n \end{vmatrix} \qquad \text{(d)}$$

determinant of 5th order

Now generalization $4 \rightarrow n$.
Proof by mathematical induction.
http://en.wikipedia.org/wiki/Mathematical_induction
Comment:
The 2D determinant Pascal pattern representation (c)

$$\frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0 = n} = \frac{\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} +1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ +1 & 1 & 3 & 3 & 0 \\ -1 & 1 & 4 & 6 & 4 \\ +1 & 1 & 5 & 10 & 10 \end{vmatrix}} = 5!$$

Pascal's triangle determinants
of 5th order

is the **“king size” solution** of the given problem:
* beautiful, simple, visually convincing,
* **extensible, portable!**

"From a drop of water," said the writer, "a logician could infer the possibility of an Atlantic or a Niagara without having seen or heard of one or the other. So all life is a great chain, the nature of which is known whenever we are shown a single link of it. Like all other arts, the Science of Deduction and Analysis is one which can only be acquired by long and patient study nor is life long enough to allow any mortal to attain the highest possible perfection in it. Before turning to those moral and mental aspects of the matter which present the greatest difficulties, let the enquirer begin by mastering more elementary problems.”
Arthur Conan Doyle. A Study in Scarlet.
http://gutenberg.spiegel.de/doyle/studie/study.htm

Oh, more!

$$\frac{\sum_{i=1}^{n} i^6}{n} = \frac{\begin{vmatrix} (-n)^0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^3 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^4 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^5 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^6 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^3 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^4 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^5 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^6 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$= p_6(n) = \frac{1}{42}(6n^6+21n^5+21n^4+11n^2+1)$$

Plot $p_6(x)$; $x \in \mathbb{R}$ and $p_6(z)$; $z \in \leq \mathbb{C}$.

$$\lim_{k\to\infty}\frac{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ -1 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 0 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ 0 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}} =$$

$(2k+1=)$ by $(2k+1=7)$ determinants

42.612923374243529927... ???

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{6}}{1}=\frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ n^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ n^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ n^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ n^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ n^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$D(k=7,n)=\begin{vmatrix} \rho^0=1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \rho^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \rho^2 & 0 & 3 & 1 & 0 & 0 & 0 \\ \rho^3 & 0 & 1 & 6 & 1 & 0 & 0 \\ \rho^4 & 0 & 0 & 5 & 10 & 1 & 0 \\ \rho^5 & 0 & 0 & 1 & 15 & 15 & 1 \\ \rho^6 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i}i^{2}}=\frac{D(k,n)}{D(k,1)}\;;\quad \rho=n(n+1)$$

$$\sum_{i=1}^{n}(-1)^{n-i}i^{2}=\sum_{i=1}^{n}i^{1}=\frac{1}{2}n(n+1)$$

$$\lim_{k\to\infty}\binom{2k+2}{2}\frac{D(k,0)}{D(k+1,0)}=\frac{\pi^2}{2}$$

$$B_{2k}=(-1)^{k-1}\frac{D(k,0)}{2(2^{2k}-1)}\quad \text{Bernoulli numbers}$$

$$D(k=7,n)=\begin{vmatrix} \rho^0=1 & 2 & 0 & 0 & 0 & 0 & 0 \\ \rho^1 & 1 & 3 & 0 & 0 & 0 & 0 \\ \rho^2 & 0 & 4 & 4 & 0 & 0 & 0 \\ \rho^3 & 0 & 1 & 10 & 5 & 0 & 0 \\ \rho^4 & 0 & 0 & 6 & 20 & 6 & 0 \\ \rho^5 & 0 & 0 & 1 & 21 & 35 & 7 \\ \rho^6 & 0 & 0 & 0 & 8 & 56 & 56 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^3}=\frac{D(k,n)}{D(k,1)}\ ;\ \rho=n(n+1)$$

$$\sum_{i=1}^{n} i^3=\left(\sum_{i=1}^{n} i^1\right)^2=\frac{\rho^2}{4}$$

$$\lim_{k\to\infty}\binom{(2k+1)+3}{3}\frac{D(k,0)}{D(k+1,0)}=\frac{4}{3}\pi^2$$