Unit n°1

Supplement 1 to Unit 1, pp 7-8

First draft

Input: a; x[0]; ε	
x: = x[0]	
	y : = x
	$x := \frac{1}{2}\left(y + \frac{a}{y} \right)$
$\lvert x - y \rvert \leq \varepsilon$	
sqr(a, ε): = x	
Output: sqr(a, ε)	

Question

How the Heron algorithm reacts for

a	x[0]
pos.	pos.
pos.	neg.
neg.	pos.
neg.	neg.

???

Heron algorithm, experiment: jump in “input a”

i	“a[i]”	x[i]
0	a_1 = 2	100.0
1	2	50.01
2	2	25.024996
3	2	12.55245805

4	2	6.355894695
5	2	3.33528161
6	2	1.967465563
7	2	1.49200089
8	2	1.416241332
9	$a_2 = 87$	1.414215014
10	87	31.46622092
11	87	17.1155453
12	87	11.09932182
13	87	9.468819279
14	87	9.328435433
15	87	9.327379113
16	87	9.327379053 = sqr(87, $\varepsilon = 10^{-9}$)
17	87	9.327379053

x-robust, *a*-robust, dynamic input follower!

Experiment 2

Input is 2-periodic

$$a(t) = A_1 + A_2 * (-1)^t$$

f. e. $A_1 = 9, A_2 = -1$

Beispiel

Input $a(t) = \text{per}(a,b)$

$a = 8, b = 12$

Output im eingeschwungenen Zustand

$x[t] \rightarrow \text{per}(A,B)$

$A = 3.51975427591741242 58...$

$B = 2.89632010710814011 73\ldots$

Input	$a(t) = a$ $a(t) = b$ $a < b$	Output	$x[t] \to A$ $x[t] \to B$ $A > B$

Algorithmus

**1 Quadriere A
$A^2 = 12.388670162\ldots$

**2 Quadriere B
$B^2 = 8.388670162..$

**3 Wenn Du richtig gerechnet hast, haben beide den gleichen frac:
$C = \text{frac}(A^2) = \text{frac}(B^2) = 0.388670162\ldots$

**4 Entwickle C in einen einfachen Kettenbruch scf(C), scf(C) ist periodisch.
scf(C) = [0;per(a_1,a_2,a_3,…,a_n)]
scf(0.388670162…)=[0;per(2,1,1,2,1,13)]

**5 scf(C) ist damit Lösung z einer quadratischen Gleichung (quadratische Irrationalität). Diese erhält man aus scf(C) auf folgendem Wege:

5.1 Umformung scf(*C*) in Bruch

Beispiel

$$z = \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{13+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{13+..}}}}}}}}}}}}$$

$$= \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{13+z}}}}}} = \frac{7z+96}{18z+247}$$

General solution

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix}\cdots\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha z+\beta \\ \gamma z+\delta \end{pmatrix}$$

5.2 Transformation into a quadratic equation

For our example

$$z = \frac{7z + 96}{18z + 247} \quad \rightarrow \; z\left(18z + 247\right) = 7z + 96$$

$$\rightarrow \; z^2 + \frac{40}{3} z - \frac{16}{3} = 0$$

General solution

$$z = \frac{\alpha z + \beta}{\gamma z + \delta} \rightarrow z^2 + \frac{\delta - \alpha}{\gamma} z - \frac{\beta}{\gamma} = 0$$

$$\rightarrow z = z_1 = -\frac{\delta - \alpha}{2\gamma} + \sqrt{\left(\frac{\delta - \alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}$$

Question

How about

$$z_2 = -\frac{\delta - \alpha}{2\gamma} - \sqrt{\left(\frac{\delta - \alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}} \quad ?$$

**6 Now we have compact formula expressions for *A* and *B*

$$A = \sqrt{\text{int}(A^2) - \frac{\delta - \alpha}{2\gamma} + \sqrt{\left(\frac{\delta - \alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$B = \sqrt{\text{int}(B^2) - \frac{\delta - \alpha}{2\gamma} + \sqrt{\left(\frac{\delta - \alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}} \;,$$

nested roots of the form $\sqrt{u+\sqrt{v}}$.

** Our example, step by step

Heron input $a(t)$ = per(a,b)
a = 8, b = 12
2-periodic

Heron execution record, $x[0]$ = 100

t	a(t)	x[t]
0	8	100.00000000000000000
1	12	50.04000000000000000
2	8	25.13990407673860911 3
3	12	12.72906163511995845 1
4	8	6.83589313569392359 83
5	12	4.00309322251214666 20
6	8	3.50038754912328885 92
7	12	2.89292438477703176 78
8	8	3.52048805755555234 65
9	12	2.89645012708148589 55
10	8	3.51972629323590278 00
11	12	2.89631515076585114 06
12	8	3.51975534277832261 02
13	12	2.89632029607528388 11
14	8	3.51975423524216096 63
15	12	2.89632009990356301 83
16	8	3.51975427746820080 78
17	12	2.89632010738282249 31
18	8	3.51975427585828692 44
19	12	2.89632010709766755 16
20	8	3.51975427591966665 02
21	12	2.89632010710853939 53
22	8	3.51975427591732648 11
23	12	2.89632010710812489 44
24	8	3.51975427591741570 26
25	12	2.89632010710814069 77

26	8	3.51975427591741230 09
27	12	2.89632010710814009 52
28	8	3.51975427591741243 06
29	12	2.89632010710814011 82
30	8	3.51975427591741242 56
31	12	2.89632010710814011 73
32	8	3.51975427591741242 58
33	12	2.89632010710814011 73
34	8	3.51975427591741242 58
35	12	2.89632010710814011 73
36	8	3.51975427591741242 58
37	12	2.89632010710814011 73
38	8	3.51975427591741242 58

Heron output, transient response

$x[t] \rightarrow$ per(A,B)
A = 3.519754275917412425 8...
B = 2.896320107108140117 3…
2-periodic too

Note:

Input		Output	
	$a(t) = a$		$x[t] \rightarrow A$
	$a(t) = b$		$x[t] \rightarrow B$
	$a < b$		$A > B$

Input $a(t)$ and Output $x[t]$ are out-of-phase:
large/small Input → small/large Output.

Squaring

A^2 = 12.38867016283890824133764 2...
B^2 = 8.38867016283890824133764 2…
C = 0.38867016283890824133764 2…

Simple continued fraction of C

$$\begin{aligned}\text{scf}(C) = \;& [0;\, 2,\, 1,\, 1,\, 2,\, 1,\, 13,\, 2,\, 1,\, 1,\, 2,\, 1,\, 13,\\ & 2,\, 1,\, 1,\, 2,\, 1,\, 13,\, 2,\, 1,\, 1,\, 2,\, 1,\, 13,\, 2,\\ & 1,\, 1,\, 2,\, 1,\, 13,\, 2,\, 1,\, 1,\, 2,\, 1,\, 13,\, 2,\, 1,\\ & 1,\, 2,\, 1,\, 13,\, 2,\, 1,\, 1,\, 2,\, 1,\, 13,\, 2,\, \ldots] \\ = \;& [0;\, \text{per}(2,\, 1,\, 1,\, 2,\, 1,\, 13)]\end{aligned}$$

scf(C) → **s**imple **f**raction, transformation in matrix representation

$$\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 13\end{pmatrix}\begin{pmatrix}z\\ 1\end{pmatrix} =$$

$$\begin{pmatrix}7 & 96\\ 18 & 247\end{pmatrix}\begin{pmatrix}z\\ 1\end{pmatrix} = \begin{pmatrix}7z+96\\ 18z+247\end{pmatrix} \rightarrow\; z = \frac{7z+96}{18z+247}$$

Transformation into a quadratic equation

$$z = \frac{7z+96}{18z+247} \qquad \rightarrow\; z\left(18z+247\right) = 7z+96$$

$$\rightarrow\; z^2 + \frac{40}{3}z - \frac{16}{3} = 0$$

Solution

$$\rightarrow z_{1;2} = -\frac{20}{3} \pm \sqrt{\left(\frac{20}{3}\right)^2 + \frac{16}{3}}$$

$$z = z_1 = -\frac{20}{3} + \sqrt{\left(\frac{20}{3}\right)^2 + \frac{16}{3}}$$

$$= 0.388670162\ldots$$

Solution by insertion into the general solution formula

$\alpha = 7, \beta = 96, \gamma = 18, \delta = 247 \rightarrow$

$$z = -\frac{20}{3} + \sqrt{\left(\frac{20}{3}\right)^2 + \frac{16}{3}} = 0.388670162\ldots$$

Nested roots representation of A and B by insertion into the general solution formulas

$$A = \sqrt{\text{int}(A^2) + z} = \sqrt{\frac{8}{3}(2 + \sqrt{7})}$$

$= 3.51975427591741242 58...$

$$B = \sqrt{\text{int}(B^2) + z} = \sqrt{\frac{4}{3}(1 + 2\sqrt{7})}$$

$= 2.89632010710814011 73\ldots$

ok.

** More examples, in cooperation with Dagmar Schönfeld

1

$a = 5, b = 11$
$A = 3.467890041910158036234732\ldots$
$B = 2.454844464070919027685586\ldots$
$C = 0.02626134277963766 1\ldots$
scf(C) = [0,per(38,12,1,2,3,1,8,1,3,2,1,12)]
int(A^2) = 12, int(B^2) = 6
$\alpha = 20461, \beta = 259710, \gamma = 779130, \delta = 9889441$

$$A = \sqrt{\text{int}(A^2) - \frac{\delta-\alpha}{2\gamma} + \sqrt{\left(\frac{\delta-\alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$B = \sqrt{\text{int}(B^2) - \frac{\delta-\alpha}{2\gamma} + \sqrt{\left(\frac{\delta-\alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$\frac{\delta-\alpha}{\gamma} = \frac{9889441 - 20461}{779130}$$

$$= \frac{9868980}{779130} = \frac{38 * 259710}{3 * 259710} = \frac{38}{3}$$

$$\frac{\beta}{\gamma} = \frac{259710}{779130} = \frac{259710}{3 * 259710} = \frac{1}{3}$$

gcd(δ-α,β,γ) = 259710

$$A = \sqrt{12 - \frac{19}{3} + \sqrt{\frac{361}{9} + \frac{1}{3}}} = \sqrt{\frac{1}{3}(17 + \sqrt{364})}$$

$$= 3.467890041910158036234732...$$

$$B = \sqrt{6 - \frac{19}{3} + \sqrt{\frac{361}{9} + \frac{1}{3}}} = \sqrt{\frac{1}{3}(-1 + \sqrt{364})}$$

$$= 2.454844464070919027685586\ldots$$

ok.

$a = 3, b = 5$
$A = 2.288944289920671930119007333 5\ldots$
$B = 1.799796089105776736637154104 2\ldots$
$C = 0.239265962360449034824654655\ldots$
scf(C) = [0,per(4,5,1,1,2,1,16,1,2,1,1,5)]
int(A^2) = 5, int(B^2) = 3
α =4759, β =26520, γ = 19890, δ = 110839

$$A = \sqrt{\text{int}(A^2) - \frac{\delta-\alpha}{2\gamma} + \sqrt{\left(\frac{\delta-\alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$B = \sqrt{\text{int}(B^2) - \frac{\delta-\alpha}{2\gamma} + \sqrt{\left(\frac{\delta-\alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$\frac{\delta-\alpha}{\gamma} = \frac{110839-4759}{19890}$$

$$= \frac{106080}{19890} = \frac{16*6630}{3*6630} = \frac{16}{3}$$

$$\frac{\beta}{\gamma} = \frac{26520}{19890} = \frac{4*6630}{3*6630} = \frac{4}{3}$$

gcd(δ-α,β,γ) = 6630

$$A = \sqrt{5 - \frac{8}{3} + \sqrt{\frac{64}{9} + \frac{4}{3}}} = \sqrt{\frac{1}{3}(7 + \sqrt{76})}$$

$$= 2.288944289920671930119007333 5...$$

$$B = \sqrt{3 - \frac{8}{3} + \sqrt{\frac{64}{9} + \frac{4}{3}}} = \sqrt{\frac{1}{3}(1 + \sqrt{76})}$$

$$= 1.799796089105776736637154104 2\ldots$$

ok.

3

$a = 5, b = 7$
$A = 2.67644017286623940529731 1\ldots$
$B = 2.27229663533004983204045 1\ldots$
$C = 0.16333199893226547 06\ldots$
scf(C) = [0,per(6,8)]
int(A^2) = 7, int(B^2) = 5

$$A = \sqrt{3 + \sqrt{\frac{52}{3}}} = 2.676440172866239\ldots$$

$$B = \sqrt{1 + \sqrt{\frac{52}{3}}} = 2.272296635330049\ldots$$

4

$a = 3, b = 6$
$A = 2.542459756837412478271226\ldots$
$B = 1.861209718204199197882437\ldots$
$C = 0.46410161513775458 71\ldots$
scf(C) = [0,per(2,6)]
int(A^2) = 6, int(B^2) = 3

$$A = \sqrt{3 + \sqrt{12}} = 2.542459756837412\ldots$$
$$B = \sqrt{\sqrt{12}} = 1.861209718204199\ldots$$

5

$a = 2, b = 4$
$A = 1.519671371303185094662376\ldots$
$B = 2.075909698604085008150407\ldots$
$C = 0.30940107675850305 80\ldots$
scf(C) = [0,per(3,4)]
int(A^2) = 4, int(B^2) = 2

$$A = \sqrt{2 + \frac{4}{\sqrt{3}}} = 2.075909698604085\ldots$$
$$B = \sqrt{\frac{4}{\sqrt{3}}} = 1.519671371303185\ldots$$

$a = 4$, $b = 16$
$A = 4.352945753690325752551141\ldots$
$B = 2.635931853172732937863768\ \ldots$
$C = 0.\ 948136734570638115\ldots$
scf(C) = [0,per(1,18,3,1,1,4,4,4,1,1,3,18)]
int(A^2) = 18, int(B^2) = 6
α =83071, β = 1518660, γ = 87615, δ = 1601731

$$A = \sqrt{\text{int}(A^2) - \frac{\delta-\alpha}{2\gamma} + \sqrt{\left(\frac{\delta-\alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$B = \sqrt{\text{int}(B^2) - \frac{\delta-\alpha}{2\gamma} + \sqrt{\left(\frac{\delta-\alpha}{2\gamma}\right)^2 + \frac{\beta}{\gamma}}}$$

$$\frac{\delta-\alpha}{\gamma} = \frac{1601731-83071}{87615}$$

$$= \frac{1518660}{87615} = \frac{52*29205}{3*29205} = \frac{52}{3}$$

$$\frac{\beta}{\gamma} = \frac{1518660}{87615} = \frac{52*29205}{3*29205} = \frac{52}{3}$$

gcd(δ-α,β,γ) = 29205

$$A = \sqrt{18 - \frac{26}{3} + \sqrt{\frac{676}{9} + \frac{52}{3}}} = \sqrt{\frac{28}{3} + \frac{8}{3}\sqrt{13}}$$

$$= 4.352945753690325752551141...$$

$$B = \sqrt{6 - \frac{26}{3} + \sqrt{\frac{676}{9} + \frac{52}{3}}} = \sqrt{\frac{8}{3}(-1 + \sqrt{13})}$$

$$= 2.635931853172732937863768\ldots$$

ok.

**7 Open problem
It's your turn
Verglichen mit *a* und *b* entstehen infolge der Kettenbruchentwicklung von *C* für δ-α, β und γ sehr große Zahlenwerte, die dann aber mit einem erstaunlich großen gcd(δ-α,β,γ) wieder zu kleinen Zahlen in den nested roots für *A* und *B* reduziert werden (vgl. Beispiele 1), 2) und 6)).
Daraus erwächst die Vermutung, dass man auch auf weniger aufwendigem Wege als via Kettenbruchentwicklung und anschließender gcd-Berechnung – auf direktem Wege aus *a* und *b* – zu eben diesen Zahlen gelangen kann.

Dieser andere Weg ist gesucht!

Dazu weiteres experimentelles Material.

t	*a*(*t*)	*x*[*t*]
0	1000	100.000000000000000000000
1	2000	55.000000000000000000000
2	1000	45.681818181818181818182
3	2000	33.786182722749886928991 41
4	1000	46.490989656249527223134 28
5	2000	34.000266961326444311322 91
6	1000	46.411667252955321780412 80

7	2000	33.97898679235885137480698
8	1000	46.41944685861988457827959
9	2000	33.9810710828667452064028 6
10	1000	46.41868386440368542895632
11	2000	33.98086663679232727568243
12	1000	46.41875869598566341629761
13	2000	33.98088668783129352328417
14	1000	46.41875135678761346976873
15	2000	33.98088472129888177910228
16	1000	46.41875207658835258014082
17	2000	33.98088491416890640763992
18	1000	46.41875200599302315887897
19	2000	33.98088489525294487934028
20	1000	46.41875201291674560775561
21	2000	33.98088489710815071793717
22	1000	46.41875201223769314150789
23	2000	33.98088489692619915798584
24	1000	46.41875201230429203470759
25	2000	33.98088489694404427763549
26	1000	46.41875201229776026758224
27	2000	33.98088489694229409590910
28	1000	46.41875201229840087855533
29	2000	33.98088489694246574710200
30	1000	46.41875201229833804985814
31	2000	33.98088489694244891220333
32	1000	46.41875201229834421185873
33	2000	33.98088489694245056330641
34	1000	46.41875201229834360751305
35	2000	33.98088489694245040137247
36	1000	46.41875201229834366678499
37	2000	33.98088489694245041725434
38	1000	46.4187520122983436 6097182
39	2000	33.9808848969424504 1569671

t	$a(t)$	$x[t]$
0	1000	100.00000000000000000000
1	100	55.00000000000000000000
2	1000	28.40909090909090909090
3	100	31.80454545454545454545
4	1000	17.47437505684254290799953
5	100	37.350514092235185087866 77
6	1000	20.013926706099448647441 41
7	100	34.989567084061684683469 32
8	1000	18.923780930877525471173 48
9	100	35.883671706002627895889 65
:		
90	1000	19.216992505192648720132 81
91	100	35.627135738712288914500 35
92	1000	19.216992505192648710039 99
93	100	35.627135738712288923119 01
94	1000	19.216992505192648714009 81
95	100	35.627135738712288931972 902
96	1000	19.216992505192648712448 36
97	100	35.627135738712288921062 41

t	$a(t)$	$x[t]$
0	100	100.00000000000000000000
1	-10	50.50000000000000000000
2	100	25.150990099009900990099 01
3	-10	14.563488357250251154433 42
4	100	6.938419840571817605731 713
5	-10	10.675461653227935324536 98
6	100	4.869367006629831249700 500
7	-10	12.702958605997270198599 76
8	100	5.957870211204899746664 193
9	-10	11.371195800702111538256 91
:		
90	100	5.498880947351890333058 537
91	-10	11.842199614802399882881 57
92	100	5.498880949195993906680 750

93	-10	11.842199612675105851669 64
94	100	5.498880948056500875444067
95	-10	11.842199613989585557790 93
96	100	5.498880948760606863077929
97	-10	11.842199613177353269890 26
98	100	5.498880948325531589876237
99	-10	11.842199613679241040685 07

t	$a(t)$	$x[t]$
0	100	100.000000000000000000000
1	1	50.500000000000000000000
2	100	25.259900990099009900990 10
3	1	14.609372347082821951964 92
4	100	7.338910778686740115795410
5	1	10.482455507182006562733 18
6	100	5.288926501242011891070330
7	1	12.098177532386573012361 63
8	100	6.090417305029942709933191
9	1	11.254826728883250980967 86
:		
160	100	5.802514901479988794975777
161	1	11.518210763043769033372 24
162	100	5.802514901479988794975776
163	1	11.518210763043769033372 24
164	100	5.802514901479988794975776
165	1	11.518210763043769033372 24
166	100	5.802514901479988794975776
167	1	11.518210763043769033372 24
168	100	5.802514901479988794975776
169	1	11.518210763043769033372 24

Aus diesem Material ersieht man

$A^2 - B^2 = b - a$.

Beweis?

Hardwarerealisierung?
Anwendungen?