Unit 18

Is it possible to booster the productive power of the classical differential calculus by suitable extensions (more exhaustive, more integrative)?

An open end for further discussion.

Talk at the Sichuan University Chengdu

Yangtze Center of Mathematics

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1. Introduction, problem

The talk "Is it possible to booster the productive power of the classical differential calculus by suitable extensions (more exhaustive, more integrative)?" starts with the basic equation of the differential calculus, the magic formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (1)$$

in the graphic context of Figure 1.

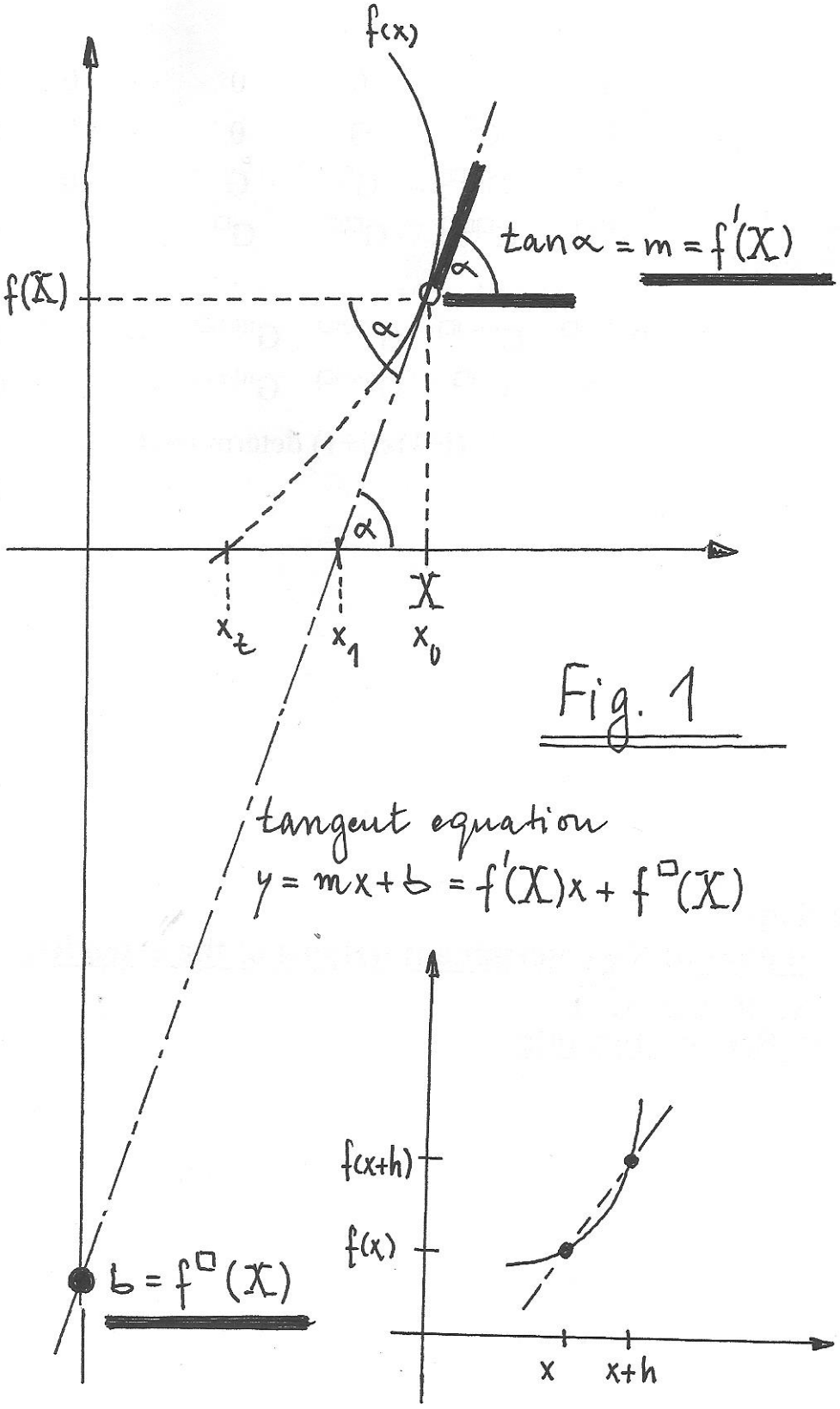


Fig. 1

Magic means here, the whole elementary differential calculus can be (re)produced from this formula.

Driven by sheer **engineering** curiosity, I want to combine in the following the magic formula (1) with tools of linear algebra (esp. determinants and matrices):

$$m = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x} \qquad (2.1)$$

$$= \lim_{h\to 0}\frac{\begin{vmatrix} f(x+h) & 1 \\ f(x) & 1 \end{vmatrix}}{\begin{vmatrix} x+h & 1 \\ x & 1 \end{vmatrix}} = \lim_{h\to 0}\frac{\begin{vmatrix} f(x) & 1 \\ f(x+h) & 1 \end{vmatrix}}{\begin{vmatrix} x & 1 \\ x+h & 1 \end{vmatrix}} = \lim_{h\to 0}\frac{\begin{vmatrix} f(x+h) & f(x) \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} x+h & x \\ 1 & 1 \end{vmatrix}} \qquad (2.2)$$

After rearrangement of equ. (2.2) we have in 2by2 matrix notation

$$\begin{pmatrix} m=f'(x) \\ b=f(x)-xf'(x)=f^{\square}(x) \end{pmatrix} = \lim_{h\to 0}\begin{pmatrix} x & 1 \\ x+h & 1 \end{pmatrix}^{-1}\begin{pmatrix} f(x) \\ f(x+h) \end{pmatrix} \qquad (3.1)$$

$$\begin{pmatrix} m=f'(x) & b=f^{\square}(x) \end{pmatrix} = \lim_{h\to 0}\begin{pmatrix} x+h & x \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} f(x+h) & f(x) \end{pmatrix} \qquad (3.2)$$

The parameters of the tangent to f(x) at (X,f(X))

→ **m=f′(X) and b=f□(X)** ← (4)

appear in the matrix equ. (3) as **equal in rank** (but asymmetric) legitimate "derivatives" of f(x) → **"prime and square"** ← with the "mother"

$f(x)= xf'(x)+f^{\square}(x)$. (5)

One gains the same impression from the determinant representations of the tangent equation

$y=mx+b = f'(X)x+f^{\square}(X)$ for f(x) at the given point (X,f(X)) (6)

as a limit

$$\lim_{h\to 0}\frac{\begin{vmatrix} 1 & 1 & 1 \\ x & X+h & X \\ y & f(X+h) & f(X) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & X+h & X \\ 0 & 1 & 1 \end{vmatrix}=h} = 0 \qquad (7.1)$$

abbreviated (see SCU YCM talk "**Computational geometry. A synoptic shortened determinant representation area, perimeter, centroid, momentum of inertia, etc. of a 2D polygon given by the Cartesian coordinates of its vertices.**")

$$\lim_{h\to 0} = \frac{\begin{vmatrix} x & X+h & X \\ y & f(X+h) & f(X) \end{vmatrix}}{\begin{vmatrix} 0 & X+h & X \\ 0 & 1 & 1 \end{vmatrix}} = 0 \qquad (7.2)$$

and using $f'(X)$ and $f^{\square}(X)$:

$$\begin{vmatrix} 1 & 1 & 1' \\ x & X & X' \\ y & f(X) & f'(X) \end{vmatrix} = 0 \; ; \; X' = 1 \; ; \; 1' = 0 \qquad (7.3)$$

$$\begin{vmatrix} 1 & 1 & 1^{\square} \\ x & X & X^{\square} \\ y & f(X) & f^{\square}(X) \end{vmatrix} = 0 \; ; \; X^{\square} = 0 \; ; \; 1^{\square} = 1 \qquad (7.4)$$

$$\begin{vmatrix} 1 & 1' & 1^{\square} \\ x & X' & X^{\square} \\ y & f'(X) & f^{\square}(X) \end{vmatrix} = 0 \qquad (7.5)$$

The talk is considering, comparing, and reflecting both “derivatives” $f'(x)$ and $f^{\square}(x)$ regarding their set of rules, cross connections, (and applications) with the title question. An engineering (metamaterial system theory)) induced question.
The talk don’t (can’t) present a definitive answer, it is an open end, it want to be a launch pad for a further discussion.

2. Set of rules for the higher “derivatives” of $f'(x)$ and $f^{\square}(x)$ (formal extension)

2.1 Cross connections between $f'(x)$ and $f^{\square}(x)$

2.1.1

$$f^{\square}(x) = f(x) - xf'(x) = \begin{vmatrix} f(x) & -x \\ -f'(x) & 1 \end{vmatrix} \tag{8.1}$$

$$f^{\square\square}(x) = f(x) - xf'(x) + x^2 f''(x) = \begin{vmatrix} f(x) & -x & 0 \\ -f'(x) & 1 & -x \\ f''(x) & 0 & 1 \end{vmatrix} \tag{8.2}$$

$$f^{\square\square\square}(x) = f(x) - xf'(x) - x^3 f'''(x)$$

$$= \begin{vmatrix} f(x) & -x & 0 & 0 \\ -f'(x) & 1 & -x & 0 \\ 0 & 0 & 1 & -x \\ -f'''(x) & 0 & 0 & 1 \end{vmatrix} \tag{8.3}$$

$$f^{(n\square)}(x) = \sum_{i=0}^{n} a(i,n) x^i f^{(i)}(x) \tag{8.4}$$

$a(0,n) = 1$; n=0,1,2,3,...

$a(1,n) = -1$; n=1,2,3,4,

$a(i,n+1) = -a(i-1,n) - (i-1)a(i,n)$; i=2(1)n ; n=1,2,3,4,...

$a(n,n) = (-1)^n$; n=0,1,2,3..

1

1 -1

1 -1 1

1 -1 0 -1

1 -1 1 2 1

1 -1 0 -5 -5 1

$$f^{\square}(x)=\begin{vmatrix} f(x) & x \\ f'(x) & 1 \end{vmatrix} \tag{8.5}$$

$$\begin{vmatrix} f^{\square}(x) & f(x) \\ f^{\square\square}(x) & f^{\square}(x) \end{vmatrix}=\begin{vmatrix} f(x) & 0 & x^2 \\ f'(x) & f(x) & x \\ f''(x) & f'(x) & 0 \end{vmatrix} \tag{8.6}$$

2.1.2

$$f'(x)=\frac{1}{x}(f(x)-f^{\square}(x)) \tag{9.1}$$

$$f''(x)=\frac{1}{x^2}(f^{\square\square}(x)-f^{\square}(x)) \tag{9.2}$$

$$f'''(x)=\frac{1}{x^3}(f^{\square}(x)-f^{\square\square\square}(x)) \tag{9.3}$$

2.1.3

$$(f'(x))^{\square}=f'(x)-xf''(x)\ =\frac{1}{x}(f(x)-f^{\square\square}(x)) \tag{10.1}$$

$$(f^{\square}(x))'=-xf''(x)\qquad =\frac{1}{x}(f^{\square}(x)-f^{\square\square}(x))\neq(f'(x))^{\square} \tag{10.2}$$

$$(f'(x))^{\square}-(f^{\square}(x))'=f'(x)=\frac{1}{x}(f(x)-f^{\square}(x)) \tag{10.3}$$

2.2 Linearity

$$(af(x)+bg(x))'=af'(x)+bg'(x) \tag{11.1}$$

$$(af(x)+bg(x))^{\square}=af^{\square}(x)+bg^{\square}(x) \tag{11.2}$$

2.3 Product rules

$$(f(x)g(x))'=f(x)g'(x)+f'(x)g(x) \tag{12.1}$$

$$(f(x)g(x))^{\square}+(f(x)g(x))=f(x)g^{\square}(x)+f^{\square}(x)g(x) \tag{12.2}$$

$$(f(x)g(x)h(x))'=f(x)g(x)h'(x)+f(x)g'(x)h(x)+f'(x)g(x)h(x) \tag{12b.1}$$

$$(f(x)g(x)h(x))^{\square}-2f(x)g(x)h(x)=f(x)g(x)h^{\square}(x)+f(x)g^{\square}(x)h(x)+f^{\square}(x)g(x)h(x) \tag{12b.2}$$

2.4 Quotient rules

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f(x)'g(x)-f(x)g'(x)}{(g(x))^2} \tag{14.1}$$

$$\frac{\left(\frac{f(x)}{x}\right)'}{\left(\frac{1}{x}\right)'} = f^{\square}(x) \tag{14.2}$$

$$\frac{\left(\frac{x}{f(x)}\right)'}{\left(\frac{1}{f(x)}\right)'} = -\frac{f^{\square}(x)}{f'(x)} \tag{14.3}$$

$$\left(\frac{f(x)}{g(x)}\right)^{\square} - \left(\frac{f(x)}{g(x)}\right) = \frac{f(x)^{\square}g(x)-f(x)g^{\square}(x)}{(g(x))^2} \tag{14.4}$$

2.5 Extended product rules

2.5.1

$$(f(x)g(x))^{(n)} = \sum_{i=0}^{n}\binom{n}{i} f^{(i)}(x)g^{(n-i)}(x) \quad \text{Leibniz Identity for /} \tag{15.1}$$

in matrix representation (n=4)

$$\begin{pmatrix} f(x)g(x) \\ (fg)' \\ (fg)'' \\ (fg)''' \\ (fg)'''' \end{pmatrix} = \begin{pmatrix} 1f & 0 & 0 & 0 & 0 \\ 1f' & 1f & 0 & 0 & 0 \\ 1f'' & 2f' & 1f & 0 & 0 \\ 1f''' & 3f'' & 3f' & 1f & 0 \\ 1f'''' & 4f''' & 6f'' & 4f' & 1f \end{pmatrix} \begin{pmatrix} g \\ g' \\ g'' \\ g''' \\ g'''' \end{pmatrix} \tag{15.2}$$

$$= \begin{pmatrix} 1g & 0 & 0 & 0 & 0 \\ 1g' & 1g & 0 & 0 & 0 \\ 1g'' & 2g' & 1g & 0 & 0 \\ 1g''' & 3g'' & 3g' & 1g & 0 \\ 1g'''' & 4g''' & 6g'' & 4g' & 1g \end{pmatrix} \begin{pmatrix} f \\ f' \\ f'' \\ f''' \\ f'''' \end{pmatrix} \tag{15.3}$$

2.5.

$$\sum_{i=0}^{n}\binom{n}{i}(f(x)g(x))^{(i\square)}=\sum_{i=0}^{n}\binom{n}{i}f^{(i\square)}(x)g^{((n-i)\square)}(x) \quad \text{"LI" for } \square \qquad (16.1)$$

in matrix representation (n=4)

$$\begin{pmatrix} f(x)g(x) \\ (fg)^{\square} \\ (fg)^{(2\square)} \\ (fg)^{(3\square)} \\ (fg)^{(4\square)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1f & 0 & 0 & 0 & 0 \\ 1f^{\square} & 1f & 0 & 0 & 0 \\ 1f^{(2\square)} & 2f^{\square} & 1f & 0 & 0 \\ 1f^{(3\square)} & 3f^{(2\square)} & 3f^{\square} & 1f & 0 \\ 1f^{(4\square)} & 4f^{(3\square)} & 6f^{(2\square)} & 4f^{\square} & 1f \end{pmatrix} \begin{pmatrix} g \\ g^{\square} \\ g^{(2\square)} \\ g^{(3\square)} \\ g^{(4\square)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1g & 0 & 0 & 0 & 0 \\ 1g^{\square} & 1g & 0 & 0 & 0 \\ 1g^{(2\square)} & 2g^{\square} & 1g & 0 & 0 \\ 1g^{(3\square)} & 3g^{(2\square)} & 3g^{\square} & 1g & 0 \\ 1g^{(4\square)} & 4g^{(3\square)} & 6g^{(2\square)} & 4g^{\square} & 1g \end{pmatrix} \begin{pmatrix} f \\ f^{\square} \\ f^{(2\square)} \\ f^{(3\square)} \\ f^{(4\square)} \end{pmatrix}$$

(16.2;3)

2.5.3

Using the normalized derivative operator notation

$$F^{(i)}=\frac{f^{(i)}(x)}{i!} \;;\quad F^{(i\square)}=\frac{f^{(\square)}(x)}{i!} \qquad (17.1;2)$$

we get for the equs (15) and (16), respectively, the reduced forms without numeric coefficients

$$(FG)^{(n)}=\sum_{i=0}^{n}F^{(i)}G^{(n-i)} \quad \text{normalized Leibniz Identity for /} \qquad (18.1)$$

in matrix representation (n=4)

$$\begin{pmatrix} FG \\ (FG)' \\ (FG)'' \\ (FG)''' \\ (FG)'''' \end{pmatrix} = \begin{pmatrix} F & 0 & 0 & 0 & 0 \\ F' & F & 0 & 0 & 0 \\ F'' & F' & F & 0 & 0 \\ F''' & F'' & F' & F & 0 \\ F'''' & F''' & F'' & F' & F \end{pmatrix} \begin{pmatrix} G \\ G' \\ G'' \\ G''' \\ G'''' \end{pmatrix} \qquad (18.2)$$

$$= \begin{pmatrix} G & 0 & 0 & 0 & 0 \\ G' & G & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 \\ G''' & G'' & G' & G & 0 \\ G'''' & G''' & G'' & G' & G \end{pmatrix} \begin{pmatrix} F \\ F' \\ F'' \\ F''' \\ F'''' \end{pmatrix} \qquad (18.3)$$

2.5.4

$$\sum_{i=0}^{n} \frac{1}{(n-i)!} (FG)^{(i\square)} = \sum_{i=0}^{n} F^{(i\square)} G^{((n-i)\square)} \quad \text{"nLI" for } \square \qquad (19.1)$$

in matrix representation (n=4)

$$\begin{pmatrix} FG \\ (FG)^{\square} \\ (FG)^{(2\square)} \\ (FG)^{(3\square)} \\ (FG)^{(4\square)} \end{pmatrix} = \begin{pmatrix} 0\downarrow & 0 & 0 & 0 & 0 \\ 1\downarrow & 0\downarrow & 0 & 0 & 0 \\ 2\downarrow & 1\downarrow & 0\downarrow & 0 & 0 \\ 3\downarrow & 2\downarrow & 1\downarrow & 0\downarrow & 0 \\ 4\downarrow & 3\downarrow & 2\downarrow & 1\downarrow & 0\downarrow \end{pmatrix}^{-1} \begin{pmatrix} F & 0 & 0 & 0 & 0 \\ F^{\square} & F & 0 & 0 & 0 \\ F^{(2\square)} & F^{\square} & F & 0 & 0 \\ F^{(3\square)} & F^{(2\square)} & F^{\square} & F & 0 \\ F^{(4\square)} & F^{(3\square)} & F^{(2\square)} & F^{\square} & F \end{pmatrix} \begin{pmatrix} G \\ G^{\square} \\ G^{(2\square)} \\ G^{(3\square)} \\ G^{(4\square)} \end{pmatrix}$$

$$= \begin{pmatrix} 0\downarrow & 0 & 0 & 0 & 0 \\ 1\downarrow & 0\downarrow & 0 & 0 & 0 \\ 2\downarrow & 1\downarrow & 0\downarrow & 0 & 0 \\ 3\downarrow & 2\downarrow & 1\downarrow & 0\downarrow & 0 \\ 4\downarrow & 3\downarrow & 2\downarrow & 1\downarrow & 0\downarrow \end{pmatrix}^{-1} \begin{pmatrix} G & 0 & 0 & 0 & 0 \\ G^{\square} & G & 0 & 0 & 0 \\ G^{(2\square)} & G^{\square} & G & 0 & 0 \\ G^{(3\square)} & G^{(2\square)} & G^{\square} & G & 0 \\ G^{(4\square)} & G^{(3\square)} & G^{(2\square)} & G^{\square} & F \end{pmatrix} \begin{pmatrix} F \\ F^{\square} \\ F^{(2\square)} \\ F^{(3\square)} \\ F^{(4\square)} \end{pmatrix}$$

$$i\downarrow = \frac{1}{i!} \qquad (19.2;3)$$

2.5 Extended quotient rule, normalized derivative operator notation equ. (17), determinant representation

2.5.1

$$\frac{d^n\left(\frac{f(x)}{g(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{f(x)}{g(x)}\right) = \left(\frac{F}{G}\right)^{(n)}$$

$$= \frac{(-1)^n}{G^{n+1}}\begin{vmatrix} F & G & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ F' & G' & G & 0 & 0 & 0 & \cdots & 0 & 0 \\ F'' & G'' & G' & G & 0 & 0 & \cdots & 0 & 0 \\ F''' & G''' & G'' & G' & G & 0 & \cdots & 0 & 0 \\ F^{(4)} & G^{(4)} & G''' & G'' & G' & G & \ddots & 0 & 0 \\ F^{(5)} & G^{(5)} & G^{(4)} & G''' & G'' & G' & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & G & 0 \\ F^{(n-1)} & G^{(n-1)} & G^{(n-2)} & G^{(n-3)} & G^{(n-4)} & G^{(n-5)} & \cdots & G' & G \\ F^{(n)} & G^{(n)} & G^{(n-1)} & G^{(n-2)} & G^{(n-3)} & G^{(n-4)} & \cdots & G'' & G' \end{vmatrix}$$

(n+1)x(n+1) determinant (20.1)

$$= \frac{1}{(-G)^{n+1}} *$$

$$* \left(\begin{vmatrix} F' & G & 0 & 0 & 0 & 0 \\ F'' & G' & G & 0 & 0 & 0 \\ F''' & G'' & G' & G & 0 & 0 \\ F^{(4)} & G''' & G'' & G' & G & 0 \\ U^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ F^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix} G - F \begin{vmatrix} G' & G & 0 & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 & 0 \\ G''' & G'' & G' & G & 0 & 0 \\ G^{(4)} & G''' & G'' & G' & G & 0 \\ G^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ G^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix} \right)$$

$detA_n$ nxn determinants $detB_n$

$$= \frac{1}{(-G)^{n+1}}((\det A_n)G - F(\det B_n)) = \frac{(-1)^n}{G^{n+1}}\begin{vmatrix} F & G \\ \det A_n & \det B_n \end{vmatrix} \tag{20.2}$$

This formulae cast a long constructivist shadow, see part 3, applications.

Special cases

$$\frac{D^{(n)}\left(\frac{f(x)}{g(x)}\right)}{D^{(n)}\left(\frac{1}{g(x)}\right)} = \frac{\left(\frac{F}{G}\right)^{(n)}}{\left(\frac{1}{G}\right)^{(n)}} = F - G\frac{\begin{vmatrix} F' & G & 0 & 0 & 0 & 0 \\ F'' & G' & G & 0 & 0 & 0 \\ F''' & G'' & G' & G & 0 & 0 \\ F^{(4)} & G''' & G'' & G' & G & 0 \\ F^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ F^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix}}{\begin{vmatrix} G' & G & 0 & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 & 0 \\ G''' & G'' & G' & G & 0 & 0 \\ G^{(4)} & G''' & G'' & G' & G & 0 \\ G^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ G^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix}}$$

$$= F - G\frac{\det A_n}{\det B_n} \tag{21.1}$$

If $g(a) = 0$ and $f(a) \neq 0$ then $\lim_{x \to a} \frac{D^{(n)}\left(\frac{f(x)}{g(x)}\right)}{D^{(n)}\left(\frac{1}{g(x)}\right)} = f(a)$. (21.2)

$$\frac{\left(\frac{x}{g(x)}\right)^{(n)}}{\left(\frac{1}{g(x)}\right)^{(n)}} = x - G\frac{\begin{vmatrix} G' & G & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 \\ G''' & G'' & G' & G & 0 \\ G^{(4)} & G''' & G'' & G' & G \\ G^{(n-1)} & G^{(n-2)} & \cdots & & G' \end{vmatrix}}{\begin{vmatrix} G' & G & 0 & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 & 0 \\ G''' & G'' & G' & G & 0 & 0 \\ G^{(4)} & G''' & G'' & G' & G & 0 \\ G^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ G^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix}}$$

$$= x - G\frac{detB_{n-1}}{detB_n} = \frac{\begin{vmatrix} x & g(x) \\ detB_{n-1} & detB_n \end{vmatrix}}{detB_n} \qquad (22)$$

$$\frac{\left(\frac{f(x)}{x}\right)^{(n)}}{\left(\frac{1}{x}\right)^{(n)}} = \begin{vmatrix} F & x & 0 & 0 & \cdots & 0 \\ F' & 1 & x & 0 & \cdots & 0 \\ F'' & 0 & 1 & x & & 0 \\ F''' & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & x \\ F^{(n)} & 0 & 0 & 0 & & 1 \end{vmatrix} = \sum_{i=0}^{n} (-x)^i F^{(i)} \qquad (23)$$

2.5.2

$$\left(\frac{F}{G}\right)^{(n\square)} = \sum_{i=0}^{n} \frac{1}{(n\text{-}i)!}\frac{(-1)^i}{G^{i+1}} \begin{vmatrix} F & G & 0 & 0 & \cdots & 0 & 0 \\ F^{\square} & G^{\square} & G & 0 & \cdots & 0 & 0 \\ F^{\square\square} & G^{\square\square} & G^{\square} & G & & 0 & 0 \\ F^{\square\square\square} & G^{\square\square\square} & G^{\square\square} & G^{\square} & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ F^{((i-1)\square)} & G^{((i-1)\square)} & G^{((i-2)\square)} & G^{((i-3)\square)} & \cdots & G^{\square} & G \\ F^{(i\square)} & G^{(i\square)} & G^{((i-1)\square)} & G^{((i-2)\square)} & \cdots & G^{\square\square} & G^{\square} \end{vmatrix}$$

(i+1)x(i+1) determinant

(24)

3. Applications

3.1 In the shadow of the determinant pattern of the extended quotient rule equ. (20)

3.1.1 Faa di Bruno's formula

$$\frac{d^n\left(f(g(x))\right)}{n!dx^n} = D^{(n)}\left(f(g(x))\right) = \begin{vmatrix} G' & G & 0 & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 & 0 \\ G''' & G'' & G' & G & 0 & 0 \\ G^{(4)} & G''' & G'' & G' & G & 0 \\ G^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ G^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix}$$

$$= \det B_n \qquad (25.1)$$

with the substitution

$$G^i \Rightarrow (-1)^i F^{(n-i)} = \left(\frac{d^{n-i}f(y)}{(n\text{-}i)!dy^{n-i}}|y=g(x)\right) ;\ i=0(1)n\text{-}1 \qquad (25.2)$$

and

$$G^{(i)} = \frac{d^i g(x)}{i!dx^i} ;\ i=1(1)n \qquad (25.3)$$

Example n=4

$$\frac{d^4\left(f(g(x))\right)}{4!dx^4} = D^{(4)}\left(f(g(x))\right) = \begin{vmatrix} G' & G & 0 & 0 \\ G'' & G' & G & 0 \\ G''' & G'' & G' & G \\ G^{(4)} & G''' & G'' & G' \end{vmatrix} = \det B_4$$

$$= G^0G'G'G'G' - 3G^1G'G'G'' + 2G^2G'G''' + G^2G''G'' - G^3G^{(4)}$$

with the substitution

$$G^i \Rightarrow (-1)^i F^{(n-i)} = \left(\frac{d^{n-i}f(y)}{(n-i)!dy^{n-i}}\bigg|_{y=g(x)} \right) ; \; i=0(1)n-1$$

$$= F'G^{(4)} + 2F''G'G'' + F''G''G'' + 3F'''G'G'G'' + F^{(4)}G'G'G'G' \qquad (26)$$

http://mathworld.wolfram.com/FaadiBrunosFormula.html
http://en.wikipedia.org/wiki/Fa%C3%A0_di_Bruno's_formula
Johnson, W.P. The Curious History of Faà di Bruno's Formula. American Mathematical Monthly **109** (2002), 217–234.
http://www.maa.org/news/monthly217-234.pdf

How about $(f(g(x)))^{(n\square)}$?

3.1.2 An extension of Newton's root finding algorithm

For the special case equ. (22) of the extended quotient rule we obtain with equ. (21.2) an extended version of Newton's recursive root finder for g(x) with the loop body

$$x := \frac{\left(\frac{x}{g(x)}\right)^{(n)}}{\left(\frac{1}{g(x)}\right)^{(n)}} = x - G\frac{\begin{vmatrix} G' & G & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 \\ G''' & G'' & G' & G & 0 \\ G^{(4)} & G''' & G'' & G' & G \\ G^{(n-1)} & G^{(n-2)} & \cdots & & G' \end{vmatrix}}{\begin{vmatrix} G' & G & 0 & 0 & 0 & 0 \\ G'' & G' & G & 0 & 0 & 0 \\ G''' & G'' & G' & G & 0 & 0 \\ G^{(4)} & G''' & G'' & G' & G & 0 \\ G^{(5)} & G^{(4)} & G''' & G'' & G' & G \\ G^{(n)} & G^{(n-1)} & & \cdots & & G' \end{vmatrix}}$$

$$= x - G\frac{detB_{n-1}}{detB_n} = \frac{\begin{vmatrix} x & g(x) \\ detB_{n-1} & detB_n \end{vmatrix}}{detB_n} \qquad (27)$$

Problem: starting value x_0, convergence.
Level n=1 (Newton's root finder)

$$x := \frac{\begin{vmatrix} x & G \\ 1 & G' \end{vmatrix}}{\begin{vmatrix} 1 & G \\ 0 & G' \end{vmatrix}} = x - \frac{G}{G'} = x - \frac{g(x)}{g'(x)} = -\frac{g^{\square}(x)}{g'(x)} \qquad (28.1.1)$$

in mathematical notation

$$x_{n+1} = -\frac{g^{\square}(x_n)}{g'(x_n)} \rightarrow x_{n+1}g'(x_n) + g^{\square}(x_n) = 0 \qquad (28.1.2)$$

and finally in determinant representation

$$\begin{vmatrix} 1 & 1' & 1^{\square} \\ x_{n+1} & x'_n & x^{\square}_n \\ y_{n+1} = 0 & g'(x_n) & g^{\square}(x_n) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ x_{n+1} & 1 & 0 \\ 0 & g'(x_n) & g^{\square}(x_n) \end{vmatrix} = 0 \qquad (28.1.3)$$

(see equ. (7)).

n=2

$$x := \frac{\begin{vmatrix} x & G & 0 \\ 1 & G' & G \\ 0 & G'' & G' \end{vmatrix}}{\begin{vmatrix} 1 & G & 0 \\ 0 & G' & G \\ 0 & G'' & G' \end{vmatrix}} = x - \frac{GG'}{\begin{vmatrix} G' & G \\ G'' & G' \end{vmatrix}} = x - \frac{1}{\frac{G'}{G} - \frac{G''}{G'}} \qquad (28.2)$$

n=3

$$x := \frac{\begin{vmatrix} x & G & 0 & 0 \\ 1 & G' & G & 0 \\ 0 & G'' & G' & G \\ 0 & G''' & G'' & G' \end{vmatrix}}{\begin{vmatrix} 1 & G & 0 & 0 \\ 0 & G' & G & 0 \\ 0 & G'' & G' & G \\ 0 & G''' & G'' & G' \end{vmatrix}} = x - \frac{G\begin{vmatrix} G' & G \\ G'' & G' \end{vmatrix}}{\begin{vmatrix} G' & G & 0 \\ G'' & G' & G \\ G''' & G'' & G' \end{vmatrix}} = x - \frac{1}{\frac{G'}{G} - \frac{\begin{vmatrix} G'' & G \\ G''' & G' \end{vmatrix}}{\begin{vmatrix} G' & G \\ G'' & G' \end{vmatrix}}} \qquad (28.3)$$

Beispiel

$g(x) = x^3 - 10$; $g'(x) = 3x^2$; $g''(x) = 6x$; $x_0 = 3$

run	x, n=1	x, n=2
start	3	3
1	2.37037037	2.203125
2	2.173508633	2.154450716
3	2.154601586	2.15443469
4	2.154434703	2.15443469
5	2.15443469	
6	2.15443469	

(28.4)

3.1.3 How about the special cases f(x) = g'(x) and g(x) = f'(x) ? Interpretation of

3.1.3.1 f(x) = g'(x)

$$\frac{d^n\left(\frac{g'(x)}{g(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{g'(x)}{g(x)}\right) = \left(\frac{G'}{G}\right)^{(n)}$$

$$= \frac{(-1)^n}{G^{n+1}} \begin{vmatrix} 1G' & G & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2G'' & G' & G & 0 & 0 & 0 & 0 & 0 & 0 \\ 3G''' & G'' & G' & G & 0 & 0 & 0 & 0 & 0 \\ 4G^{(4)} & G''' & G'' & G' & G & 0 & 0 & 0 & 0 \\ 5G^{(5)} & G^{(4)} & G''' & G'' & G' & G & 0 & 0 & 0 \\ 6G^{(6)} & G^{(5)} & G^{(4)} & G''' & G'' & G' & G & 0 & 0 \\ & & & & & & G' & G & 0 \\ nG^{(n)} & G^{(n-1)} & G^{(n-2)} & G^{(n-3)} & G^{(n-4)} & G^{(n-5)} & & G' & G \\ (n+1)G^{(n+1)} & G^{(n)} & G^{(n-1)} & G^{(n-2)} & G^{(n-3)} & G^{(n-4)} & & G'' & G' \end{vmatrix}$$

(29.1)

3.1.3.2 g(x) =f'(x)

$$\frac{d^n\left(\frac{f(x)}{f'(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{f(x)}{f'(x)}\right) = \left(\frac{F}{F'}\right)^{(n)} = \frac{(-1)^n}{(F')^{n+1}} *$$

$$* \begin{vmatrix}
F & 1F' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
F' & 2F'' & 1F' & 0 & 0 & 0 & 0 & 0 & 0 \\
F'' & 3F''' & 2F'' & 1F' & 0 & 0 & 0 & 0 & 0 \\
F''' & 4F^{(4)} & 3F''' & 2F'' & 1F' & 0 & 0 & 0 & 0 \\
F^{(4)} & 5F^{(5)} & 4F^{(4)} & 3F''' & 2F'' & 1F' & 0 & 0 & 0 \\
F^{(5)} & 6F^{(6)} & 5F^{(5)} & 4F^{(4)} & 3F''' & 2F'' & 1F' & 0 & 0 \\
 & & & & & & 2U'' & 1U' & 0 \\
F^{(n-1)} & nF^{(n)} & (n-1)F^{(n-1)} & & & & & 2F'' & 1F' \\
F^{(n)} & (n+1)F^{(n+1)} & nF^{(n)} & & & & & 3F''' & 2F''
\end{vmatrix}$$

(29.2)

3.1.4 Taylor series representation of fractional functions

$f(x) = \frac{u(x)}{v(x)}$ **at a=0 ; v(t) is normalized to v(0) = 1**

$$f(x) = \frac{u(x)}{v(x)} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} F^{(n)}(0) x^n \qquad (30)$$

$F^{(n)}(0)$ see equ. (20) with x=0 .

For the special case v(0) = V(0) = 1 (31.1)
we obtain with equ. (20) for f(x) the infinite determinant representations

$$f(x) = -\begin{vmatrix} 0 & x^0 & x^1 & x^2 & x^3 & x^4 & \rightarrow \\ U(0) & 1 & 0 & 0 & 0 & 0 & \\ U'(0) & V'(0) & 1 & 0 & 0 & 0 & \\ U''(0) & V''(0) & V'(0) & 1 & 0 & 0 & \\ U'''(0) & V'''(0) & V''(0) & V'(0) & 1 & 0 & \\ U^{(4)}(0) & V^{(4)}(0) & V'''(0) & V''(0) & V'(0) & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & U(0) & U'(0) & U''(0) & U'''(0) & U^{(4)}(0) & \rightarrow \\ x^0 & 1 & V'(0) & V''(0) & V'''(0) & V^{(4)}(0) & \\ x^1 & 0 & 1 & V'(0) & V''(0) & V'''(0) & \\ x^2 & 0 & 0 & 1 & V'(0) & V''(0) & \\ x^3 & 0 & 0 & 0 & 1 & V'(0) & \\ x^4 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (31.2)$$

3.1.5 Beispiele zu 3.1.4

3.1.5.1

$u(x)=\sin x$; $v(x)=(\sin x)'=\cos x$; $f(x)=\tan x=\dfrac{\sin x}{\cos x}$ (32.1)

$$\tan x = \left|\frac{1}{1!}\right| x + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^5 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^7 + \ldots$$

$$= \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

(32.2)

3.1.5.2 For comparision

$u(x)=\sinh x$; $v(x)=(\sinh x)'=\cosh x$; $f(x)=\tanh x=\dfrac{\sinh x}{\cosh x}$ (32.3)

$$\tanh x = -\begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (32.4)$$

3.1.5.3

u(x)=1, v(x)=cos x;

$$\frac{1}{\cos x} = 1 + \left|\frac{1}{2!}\right| x^2 + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^4 + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^6 + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^8 + \ldots$$

$$= \begin{vmatrix} +x^0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{1385}{40320}x^8 + \ldots$$

(33)

3.1.5.4

$$u(x)=1,\ v(x)=\frac{\sin x}{x}\ ;\ f(x)=\frac{1}{\frac{\sin x}{x}}$$

$$\frac{x}{\sin x}=1+\left|\frac{1}{3!}\right|x^2+\begin{vmatrix}\frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^4+\begin{vmatrix}\frac{1}{3!} & \frac{1}{1!} & 0\\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^6+\begin{vmatrix}\frac{1}{3!} & \frac{1}{1!} & 0 & 0\\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0\\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^8+\ldots$$

$$=\begin{vmatrix}+x^0 & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \rightarrow\\ -x^2 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \\ +x^4 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \\ -x^6 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{1!} & \\ \downarrow & & & & & & \searrow\end{vmatrix}$$

$$=1+\frac{1}{6}x^2+\frac{7}{360}x^4+\frac{31}{15120}x^6+\frac{127}{604800}x^8+\ldots$$

(34)

3.1.5.5 For comparison

$$e^{x} = 1+\left|\frac{1}{1!}\right|x+\begin{vmatrix}\frac{1}{1!} & \frac{1}{0!}\\ \frac{1}{2!} & \frac{1}{1!}\end{vmatrix}x^{2}+\begin{vmatrix}\frac{1}{1!} & \frac{1}{0!} & 0\\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!}\\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!}\end{vmatrix}x^{3}+\begin{vmatrix}\frac{1}{1!} & \frac{1}{0!} & 0 & 0\\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0\\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!}\\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!}\end{vmatrix}x^{4}+\ldots$$

$$= 1+\frac{1}{1!}x+\frac{1}{2!}x^{2}+\frac{1}{3!}x^{3}+\frac{1}{4!}x^{4}+\ldots$$

.

$$= \begin{vmatrix} +x^{0} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ -x^{1} & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ +x^{2} & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ -x^{3} & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ +x^{4} & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

(35)

3.1.5.6

$$\frac{(\sin x)^{\square}}{(\sin x)'^{\square}} = \frac{\sin x - x\cos x}{\cos x + x\sin x} = -\begin{vmatrix} 0 & \frac{2}{3!} & \frac{-4}{5!} & \frac{6}{7!} & \frac{-8}{9!} & \frac{10}{11!} & \rightarrow \\ x^3 & 1 & \frac{1}{2!} & \frac{-3}{4!} & \frac{5}{6!} & \frac{-7}{8!} & \\ x^5 & 0 & 1 & \frac{1}{2!} & \frac{-3}{4!} & \frac{5}{6!} & \\ x^7 & 0 & 0 & 1 & \frac{1}{2!} & \frac{-3}{4!} & \\ x^9 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^{11} & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (36.1)$$

$$\frac{(\sin x)^{\square\square}}{(\sin x)'^{\square\square}} = \frac{\sin x - x\cos x - x^2 \sin x}{\cos x + x\sin x - x^2 \cos x}$$

$$= -\begin{vmatrix} 0 & \frac{-4}{3!} & \frac{16}{5!} & \frac{-36}{7!} & \frac{64}{9!} & \frac{-100}{11!} & \rightarrow \\ x^3 & 1 & \frac{-1}{2!} & \frac{9}{4!} & \frac{-25}{6!} & \frac{49}{8!} & \\ x^5 & 0 & 1 & \frac{-1}{2!} & \frac{9}{4!} & \frac{-25}{6!} & \\ x^7 & 0 & 0 & 1 & \frac{-1}{2!} & \frac{9}{4!} & \\ x^9 & 0 & 0 & 0 & 1 & \frac{3}{2!} & \\ x^{11} & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (36.2)$$

3.1.5.7

$$\frac{\sin x}{\cosh x} = -\begin{vmatrix} 0 & \frac{1}{1!} & \frac{-1}{3!} & \frac{1}{5!} & \frac{-1}{9!} & \frac{1}{7!} & \rightarrow \\ x^1 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^3 & 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^5 & 0 & 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \\ x^7 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^9 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \tag{37}$$

3.1.5.8

$$\frac{\cos x}{\cosh x} = -\begin{vmatrix} 0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} & \rightarrow \\ x^0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^2 & 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \tag{38.1}$$

$$\frac{\cosh x}{\cos x} = -\begin{vmatrix} 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \rightarrow \\ x^0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} & \\ x^2 & 0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{-1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \tag{38.2}$$

$$\frac{(\cosh x)^{\square}}{(\cos x)^{\square}} = \frac{\cosh x - x\sinh x}{\cos x + x\sin x} = -\begin{vmatrix} 0 & 1 & \frac{-1}{2!} & \frac{-3}{4!} & \frac{-5}{6!} & \frac{-7}{8!} & \rightarrow \\ x^0 & 1 & \frac{1}{2!} & \frac{-3}{4!} & \frac{5}{6!} & \frac{-7}{8!} & \\ x^2 & 0 & 1 & \frac{1}{2!} & \frac{-3}{4!} & \frac{5}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1}{2!} & \frac{-3}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \tag{38.3}$$

$$\frac{(\cosh)^{\square\square}}{(\cos x)^{\square\square}} = \frac{\cosh x - x\sinh x + x^2\cosh x}{\cos x + x\sin x - x^2\cos x}$$

$$= -\begin{vmatrix} 0 & 1 & \frac{1}{2!} & \frac{9}{4!} & \frac{25}{6!} & \frac{49}{8!} & \rightarrow \\ x^0 & 1 & \frac{-1}{2!} & \frac{9}{4!} & \frac{-25}{6!} & \frac{49}{8!} & \\ x^2 & 0 & 1 & \frac{-1}{2!} & \frac{9}{4!} & \frac{-25}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{-1}{2!} & \frac{9}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{-1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (38.4)$$

$$\frac{(\cosh)^{\square\square}}{(\cos x)^{\square\square\square}} = \frac{\cosh x - x\sinh x + x^2\cosh x}{\cos x + x\sin x - x^3\sin x}$$

$$= -\begin{vmatrix} 0 & 1 & \frac{1}{2!} & \frac{9}{4!} & \frac{25}{6!} & \frac{49}{8!} & \rightarrow \\ x^0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \frac{125}{6!} & \frac{-343}{8!} & \\ x^2 & 0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \frac{125}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (38.5)$$

$$\frac{(\cosh)^{\square\square\square}}{(\cos x)^{\square\square\square}} = \frac{\cosh x - x\sinh x - x^3\sinh x}{\cos x + x\sin x - x^3\sin x}$$

$$= - \begin{vmatrix} 0 & 1 & \frac{-1}{2!} & \frac{-27}{4!} & \frac{-125}{6!} & \frac{-343}{8!} & \rightarrow \\ x^0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \frac{125}{6!} & \frac{-343}{8!} & \\ x^2 & 0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \frac{125}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (38.6)$$

$$\frac{(\cosh)^{\square\square/\square}}{(\cos x)^{\square\square\square}} = \frac{-2x^2\sinh x - x^3\cosh x}{\cos x + x\sin x - x^3\sin x}$$

$$= - \begin{vmatrix} 0 & 1 & \frac{-2*9}{3!} & \frac{-4*25}{5!} & \frac{-6*49}{7!} & \frac{-8*81}{9!} & \rightarrow \\ x^3 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \frac{125}{6!} & \frac{-343}{8!} & \\ x^5 & 0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \frac{125}{6!} & \\ x^7 & 0 & 0 & 1 & \frac{1}{2!} & \frac{-27}{4!} & \\ x^9 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^{11} & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (38.7)$$

The examples in 3.1.5.8 show:
The sequence of prime and square derivatives of numerator and denominator have in these cases a navigation and editing effect in the related reciprocal of the factorial determinant patterns.

3.1.5.9

$$f(x) = \frac{(1+x)^k - (1+x)^k}{(1+x)^k + (1+x)^k} = - \begin{vmatrix} 0 & \binom{k}{1} & \binom{k}{3} & \binom{k}{5} & \binom{k}{7} & \rightarrow \\ x^1 & \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \binom{k}{6} & \\ x^3 & 0 & \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \\ x^5 & 0 & 0 & \binom{k}{0} & \binom{k}{2} & \\ x^7 & 0 & 0 & 0 & \binom{k}{0} & \\ \downarrow & & & & & \searrow \end{vmatrix} \qquad (39)$$

3.1.5.10

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i} = -\begin{vmatrix} 0 & a_0 & a_1 & a_2 & a_3 & \rightarrow \\ x^0 & 1 & b_1 & b_2 & b_3 & \\ x^1 & 0 & 1 & b_1 & b_2 & \\ x^2 & 0 & 0 & 1 & b_1 & \\ x^3 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix} ; b_0 = 1 \qquad (40)$$

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Die Anwendungsfelder der Ableitung $f'(x)$ - Basis der Differentialrechnung, einer der tragenden Säulen der Mathematik - in den Naturwissenschaften und in der Technik sind Legion.
Wie steht es mit spezifischen Anwendungen von $f^{\square}(x)$?