Unit n°126

cos and nested radicals

Basis formula

$$2\begin{cases}\cos\dfrac{x}{2^k}\\ \cosh\dfrac{x}{2^k}\end{cases}=\sqrt{2+\sqrt{2+\ldots\sqrt{2+2\begin{cases}\cos x\\ \cosh x\end{cases}}}}$$

k times radicand 2

Exposition for cos, x=π/4, x=π/6

$$2\cos\left(\frac{a_0=1}{2}*\frac{\pi}{2}\right)=\sqrt{2}$$

$$2\cos\left(\frac{b_0=1}{3}*\frac{\pi}{2}\right)=\sqrt{3}$$

$$2\cos\left(\frac{a_0=1}{2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2}}$$

$$2\cos\left(\frac{a_1=3}{2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2}}$$

$$2\cos\left(\frac{b_0=1}{2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{3}}$$

$$2\cos\left(\frac{b_1=5}{2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{3}}$$

$$2\cos\left(\frac{a_0=1}{2*2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$2\cos\left(\frac{a_1=3}{2*2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2-\sqrt{2}}}$$

$$2\cos\left(\frac{a_2=7}{2*2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{2}}}$$

$$2\cos\left(\frac{a_3=5}{2*2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2-\sqrt{2}}}$$

$$2\cos\left(\frac{b_0=1}{2*2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2+\sqrt{3}}}$$

$$2\cos\left(\frac{b_1=5}{2*2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2-\sqrt{3}}}$$

$$2\cos\left(\frac{b_2=11}{2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{3}}}$$

$$2\cos\left(\frac{b_3=7}{2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2-\sqrt{3}}}$$

$$2\cos\left(\frac{a_0=1}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$2\cos\left(\frac{a_1=3}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2}}}}$$

$$2\cos\left(\frac{a_2=7}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2}}}}$$

$$2\cos\left(\frac{a_3=5}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2}}}}$$

$$2\cos\left(\frac{a_4=15}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$2\cos\left(\frac{a_5=13}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$$

$$2\cos\left(\frac{a_6=9}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2}}}}$$

$$2\cos\left(\frac{a_7=11}{2*2*2*2}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}$$

$$2\cos\left(\frac{b_0=1}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$2\cos\left(\frac{b_1=5}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{3}}}}$$

$$2\cos\left(\frac{b_2=11}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{3}}}}$$

$$2\cos\left(\frac{b_3=7}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{3}}}}$$

$$2\cos\left(\frac{b_4=23}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

$$2\cos\left(\frac{b_5=19}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{3}}}}$$

$$2\cos\left(\frac{b_6=13}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{3}}}}$$

$$2\cos\left(\frac{b_7=17}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{3}}}}$$

Problem

Continue this table and find a general formula for the relation

"argument of cos ⇔ nested radical" (see Unit 90).

Solution 1 for 22…2 ⇔

$$2*\cos\left(\frac{2\alpha+1}{2^{m+1}}*\frac{\pi}{2}\right)=$$

$$\underbrace{\sqrt{2+(-1)^{\beta_{m-1}}\sqrt{2+(-1)^{\beta_{m-2}}\sqrt{2+\ldots+(-1)^{\beta_1}\sqrt{2+(-1)^{\beta_0}\sqrt{2}}}}}}_{\text{(m+1) times radicand 2}}$$

with

binary $\alpha=\alpha_{m-1}\alpha_{m-2}\ldots\alpha_1\alpha_0=g^{-1}(\beta)$

decimal $\alpha=\sum_{i=0}^{m-1}\alpha_i 2^i$

binary $\beta=\beta_{m-1}\beta_{m-2}\ldots\beta_1\beta_0=g(\alpha)$

g – Gray code ⇔

$\beta=g(\alpha)$ – $\beta_i=\alpha_{i+1}\oplus\alpha_i$

$\alpha=g^{-1}(\beta)$ – $\alpha_i=\beta_{m-1}\oplus\beta_{m-2}\oplus\ldots\oplus\beta_i$

$\oplus$ – exclusive or

Example

m=3, binaryβ=101 → binaryα=110 → decimalα=6

2α+1=13 →

$$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}=2\cos\left(\frac{13}{2^4}*\frac{\pi}{2}\right)$$

Solution 1 for 22…23 ⇐

Find a similar solution for 22…23 ⇐

$$2*\cos\left(\frac{A}{2^m*3}*\frac{\pi}{2}\right)\Leftarrow$$

$$\underbrace{\sqrt{2+(-1)^{\beta_{m-1}}\sqrt{2+(-1)^{\beta_{m-2}}\sqrt{2+\ldots+(-1)^{\beta_1}\sqrt{2+(-1)^{\beta_0}\sqrt{3}}}}}}_{\text{m times radicand 2, followed by the radicand 3}}$$

$$A = 3\alpha + \alpha_0 + 1 = 4\alpha_0 + 3\sum_{i=1}^{m-1}\alpha_i 2^i + 1$$

with

$$\alpha = g^{-1}(\beta) \;-\; \alpha_i = \beta_{m-1} \oplus \beta_{m-2} \oplus \ldots \oplus \beta_i$$

$\oplus$ – exclusive or

Example

m=3, binaryβ=101 → binaryα=110 → A = 19 →

$$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{3}}}} = 2\cos\left(\frac{19}{2^3*3}*\frac{\pi}{2}\right)$$

Find A ⇒ β.

Solution 2

Notations

$$2\cos\left(\frac{\text{numerator}}{\text{denominator}}*\frac{\pi}{2}\right) = \text{nested radical}$$

represented by the binary sign word

$$w \in \left\{+(\triangleq 0), -(\triangleq 1)\right\}^*$$

f.i.

$$2\cos\left(\frac{19}{2*2*2*3}*\frac{\pi}{2}\right)=\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{3}}}}$$

$$\text{den}=2*2*2*3=24,\quad \text{num}=19$$

$$\text{w}=-+-\triangleq \text{bin}\beta{=}101\triangleq \text{dec}\beta{=}5$$

Analysis of the “cos, x=π/4, x=π/6” material

Recursion schema for 2…22

den	num	w	bin	dec
2*2*2*2=16	13	− + −	101	5
2=2	a_0=1	ε		
2*2=4	a_0=1	+	0	0
	a_1=den−a_0 =4−1=3	−	1	1
2*2*2=8	a_0=1	+ +	00	0
	a_1=3	+ −	01	1
	a_2=den−a_0 =8−1=7	− +	10	2
	a_3=den−a_1 =8−4+1=5	− −	11	3
2*2*2*2=16	a_0=1	+ + +	000	0
	a_1=3	+ + −	001	1
	a_2=7	+ − +	010	2
	a_3=5	+ − −	011	3
	a_4=den−a_0 =16−1=15	− + +	100	4
	a_5=den−a_1 =16−4+1=13	− + −	101	5
	a_6=den−a_2 =16−8+1=9	− − +	110	6
	a_7=den−a_3 =16−8+4−1=11	− − −	111	7

Recursion schema for 2…23

den	num	w	bin	dec
2*2*2*3=24	19	− + −	101	5
3=3	b_0=1	ε		
2*3=6	b_0=1	+	0	0
	b_1=den−b_0 =6−1=5	−	1	1
2*2*3=12	b_0=1	+ +	00	0
	b_1=5	+ −	01	1
	b_2=den−b_0 =12−1=11	− +	10	2
	b_3=den−b_1 =12−6+1=7	− −	11	3
2*2*2*3=24	b_0=1	+ + +	000	0
	b_1=5	+ + −	001	1
	b_2=11	+ − +	010	2
	b_3=7	+ − −	011	3
	b_4=den−b_0 =24−1=23	− + +	100	4
	b_5=den−b_1 =24−6+1=19	− + −	101	5
	b_6=den−b_2 =24−12+1=13	− − +	110	6
	b_7=den−b_3 =24−12+6−1=17	− − −	111	7

Find formula representations for a_i and b_i , basing on the above

recursion schemata.

Lateral problems
Problem 1
** Proof and extend

$$\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}} = 1+\sqrt{2-\sqrt{2-\sqrt{2}}}$$

$$= 1+\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{1+\frac{\sqrt{2+\sqrt{2}}}{1+\sqrt{2}}}$$

→ infinite continued fraction tower

$$1+\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{1+\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{1+\frac{\sqrt{2+\sqrt{2}}}{1+\sqrt{2}}}} \to 2$$

** Discuss the graph of the function

$$f(x) = 1 + \cfrac{\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}}}{1+\cfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{1+\cfrac{\sqrt{x+\sqrt{x}}}{1+\cfrac{\sqrt{x}}{1+\sqrt{x}}}}}$$

with domain = codomain = $\mathbb{R}, \mathbb{C}$, resp.

** Find a similar infinite continued fraction representation for $f'(x)$.

Problem 2

Calculate and discuss the values of the 22…22 and 22…23 scf

** $[a_0;a_1,a_2, a_3, a_4, a_5, a_6, a_7,\ldots] =$
$[1;3,7,5,15,13,9,11,\ldots];$
$a_i \in \underline{A}=\{x|x,k\in\infty \text{ } \underline{\text{and}} \text{ } x=2k\pm1\}$

** $[b_0;b_1,b_2, b_3, b_4, b_5, b_6, b_7,\ldots] =$
$[1;5,11,7,23,19,13,17,\ldots];$
$b_i \in \underline{B}=\{1,x|x,k\in\infty \text{ } \underline{\text{and}} \text{ } x=6k\pm1\}$

Plot the prime density in $\underline{A}$ and $\underline{B}$ resp.