Unit n°117 continuation 4

Bessel versus Tangent functions

7) Tfanou,bab.

These formulas are not only useful (fast converging, taxonomy-compatible), but also beautiful.

$$e^x = [0;1,\frac{-1}{x},-2,\frac{3}{x},2,\frac{-5}{x},-2,\frac{7}{x},2,\frac{-9}{x},-2,\frac{11}{x},...]$$

$$\frac{\ln(1+x)}{x} = [0;1,\frac{2}{1x},3,\frac{2}{2x},5,\frac{2}{3x},7,\frac{2}{4x},9,\frac{2}{5x},11,...]$$

$$\left.\begin{matrix}\tan x \\ \tanh x\end{matrix}\right\} = \cfrac{1}{\frac{1}{x} \mp \cfrac{1}{\frac{3}{x} \mp \cfrac{1}{\frac{5}{x} \mp \cfrac{1}{\frac{7}{x} \mp \cfrac{1}{\frac{9}{x} \mp \cfrac{1}{\frac{11}{x} \mp ...}}}}}} =$$

$$[0;\frac{1}{x},\mp\frac{3}{x},\frac{5}{x},\mp\frac{7}{x},\frac{9}{x},\mp\frac{11}{x},...]$$

$$\left.\begin{matrix}\arctan x\\ \operatorname{artanh} x\end{matrix}\right\} = \cfrac{1}{\cfrac{1}{x} \pm \cfrac{1^2}{\cfrac{3}{x} \pm \cfrac{2^2}{\cfrac{5}{x} \pm \cfrac{3^2}{\cfrac{7}{x} \pm \cfrac{4^2}{\cfrac{9}{x} \pm \cfrac{5^2}{\cfrac{11}{x} \pm \ldots}}}}}}$$

Find more!

8) Miscellaneous, cfchang

$$\frac{I_0(2)}{I_0(2)+I_1(2)} = 0.589006317\ldots$$

$$= [0;1,1,2,3,4,5,6,7,\ldots]$$

$$[0;1,-1,-2,3,4,-5,-6,7,8,-9,-10.11,12,\ldots]$$

$$= 2.693505525\ldots$$

$$= [2;1,2,3,1,4,5,1,6,7,1,8,9,1,\ldots]$$

$$=$$ formula expression?

$$[0;1,-1,2,-3,4,-5,6,-7,8,-9,10,-11,12,...]$$
$$= -0.634549847...$$
$$= -[0;1,1,2,1,3,1,4,1,5,1,6,1,7,...]$$
$$= \text{ formula expression?}$$

$$[0;1,-1,-2,-3,4,5,6,-7,-8,-9,10,11,12,-13,-14,-15,\ldots]$$
$$= 3.362118106\ldots$$
$$= [3;2,1,3,5,5,1,6,8,8,1,\ldots]$$

Deeper:

[0; 1, -1, 2, -3, 4, -5, 6, -7, 8, -9, 10, -11, 12, -13, 14, -15, 16, -17, 18, -19, 20, -21, 22, -23, 24, -25, 26, -27, 28, -29, 30, -31, 32, -33, 34, -35, 36, -37, 38, -39, 40, -41, 42, -43, 44, -45, 46, -47, 48, -49, 50, -51, 52, -53, 54, -55, 56, -57, 58, -59, 60, -61, 62, -63, 64, -65, 66, -67, 68, - 69, 70, -71, 72, -73, 74, -75, 76, -77, 78, -79, 80, -81, 82, -83, 84, -85, 86, -87, 88, -89, 90, -91, 92, -93, 94, -95, …] =

-0.6345498477561320902185449882754410368815899967156015814939456500996985086844746630012519428190958289… =

-[0; 1, 1, 1, 2, 1, 3, **1, 4, 1, 5, 1,** 6, 1, 7, 1, 8, 1, 9, 1, 10, 1, 11, 1, 12, 1, 13, 1, 14, 1, 15, 1, 16, 1, 17, 1, 18, 1, 19, 1, 20, 1, 21, 1, 22, 1, 23, 1, 24, 1, 25, 1, 26, 1, 27, 1, 28, 1, 29, 1, 30, 1, 31, 1, 32, 1, 33, 1, 34, 1, 35, 1, 36, 1, 37, 1, 38, 1, 39, 1, 40, 1, 41, 1, 42, 1, 43, 1, 44, 1, 45, 1, 46, 1, 47, 1, 48, 1, 49, 1,…] = ???

[0; 1, -1, -2, 3, 4, -5, -6, 7, 8, -9, -10, 11, 12, -13, -14, 15, 16, -17, -18, 19, 20, -21, -22, 23, 24, -25, -26, 27, 28, -29, -30, 31, 32, -33, -34, 35, 36, -37, -38, 39, 40, -41, -42, 43, 44, -45, -46, 47, 48, -49, -50, 51, 52, -53, -54, 55, 56, -57, -58, 59, 60, -61, -62, 63, 64, -65, -66, 67, 68, -69, -70, 71, 72, -73, -74, 75, 76, -77, -78, 79, 80, -81, -82, 83, 84, -85, -86, 87, 88, -89, -90, 91, 92, -93, -94, …] =

2.693505524975190050591839074133893936591881997432639920317818408228934456489758621726872951359375478… =

[2; 1, 2, 3, **1, 4, 5, 1, 6, 7, 1,** 8, 9, 1, 10, 11, 1, 12, 13, 1, 14, 15, 1, 16, 17, 1, 18, 19, 1, 20, 21, 1, 22, 23, 1, 24, 25, 1, 26, 27, 1, 28, 29, 1, 30, 31, 1, 32, 33, 1, 34, 35, 1, 36, 37, 1, 38, 39, 1, 40, 41, 1, 42, 43, 1, 44, 45, 1, 46, 47, 1, 48, 49, 1, 50, 51, 1, 52, 53, 1, 54, 55, 1, 56, 57, 1, 58, 59, 1, 60, 61, 1, 62, 63, 1, 64, 65, 1, …] = ???

[0; 1, -1, -2, -3, 4, 5, 6, -7, -8, -9, 10, 11, 12, -13, -14, -15, 16, 17, 18, -19, -20, -21, 22, 23, 24, -25, -26, -27, 28, 29, 30, -31, -32, -33, 34, 35, 36, -37, -38, -39, 40, 41, 42, -43, -44, -45, 46, 47, 48, -49, -50, -51, 52, 53, 54, -55, -56, -57, 58, 59, 60, -61, -62, -63, 64, 65, 66, -67, -68, -69, 70, 71, 72, -73, -74, -75, 76, 77, 78, -79, -80, -81 82, 83, 84, -85, -86, -87, 88, 89, 90, -91, -92, -93, 94, 95, …] =

3.362118106182372715726406185630136545066499643338289106822551462404836221057786899798409581394162343… =

[3; 2, 1, 3, 5, 5, **1, 6, 8, 8, 1, 9, 11, 11, 1,** 12, 14, 14, 1, 15, 17, 17, 1, 18, 20, 20, 1, 21, 23, 23, 1, 24, 26, 26, 1, 27, 29, 29, 1, 30, 32, 32, 1, 33, 35, 35, 1, 36, 38, 38, 1, 39, 41, 41, 1, 42, 44, 44, 1, 45, 47, 47, 1, 48, 50, 50, 1, 51, 53, 53, 1, 54, 56, 56, 1, 57, 59, 59, 1, 60, 62, 62, 1, 63, 65, 65, 1, 66, 68, 68, 1, 69, 71, 71, 1, 72, 74, 74, 1, 75, …] = ???

[0, 1, -1, -2, -3, -4, 5, 6, 7, 8, -9, -10, -11, -12, 13, 14, 15, 16, -17, -18, -19, -20, 21, 22, 23, 24, -25, -26, -27, -28, 29, 30, 31, 32, -33, -34, -35, -36, 37, 38, 39, 40, -41, -42, -43, -44, 45, 46, 47, 48, -49, -50, -51, -52, 53, 54, 55, 56, -57, -58, -59, -60, 61, 62, 63, 64, -65, -66, -67, -68, 69, 70, 71, 72, -73, -74, -75, -76, 77, 78, 79, 80, -81, -82, -83, -84, 85, 86, 87, 88, -89, -90, -91, -92, 93, 94, 95, 96, -97, -98, …] =

3.306492580765757099702675871742095006434099261947062046460925998468639202172569792686500723531604382… =

[3; 3, 3, 1, 4, 6, 7, 7, **1, 8, 10, 11, 11, 1, 12, 14, 15, 15, 1,** 16, 18, 19, 19, 1, 20, 22, 23, 23, 1, 24, 26, 27, 27, 1, 28, 30, 31, 31, 1, 32, 34, 35, 35, 1, 36, 38, 39, 39, 1, 40, 42, 43, 43, 1, 44, 46, 47, 47, 1, 48, 50, 51, 51, 1, 52, 54, 55, 55, 1, 56, 58, 59, 59, 1, 60, 62, 63, 63, 1, 64, 66, 67, 67, 1, 68, 70, 71, 71, 1, 72, 74, 75, 75, 1, 76, 78, 79, 79, 1, 80, …] = ???

[0; 1, -1, -2, -3, -4, -5, 6, 7, 8, 9, 10, -11, -12, -13, -14, -15, 16, 17, 18, 19, 20, -21, -22, -23, -24, -25, 26, 27, 28, 29, 30, -31, -32, -33, -34, - 35, 36, 37, 38, 39, 40, -41, -42, -43, -44, -45, 46, 47, 48, 49, 50, -51, -52, -53, -54, -55, 56, 57,

58, 59, 60, -61, -62, -63, -64, -65, 66, 67, 68, 69, 70, -71, -72, -73, -74, -75, 76, 77, 78, 79, 80, -81, -82, -83, -84, -85, 86, 87, 88, 89, 90, -91, -92, -93, -94, -95, 96, 97, 98, …]

3.308859880970481863803430749750577316130737724178102575011520162112217349109972719804630160123522 11… =

[3; 3, 4, 4, 1, 5, 7, 8, 9, 9, **1, 10, 12, 13, 14, 14, 1, 15, 17, 18, 19, 19, 1,** 20, 22, 23, 24, 24, 1, 25, 27, 28, 29, 29, 1, 30, 32, 33, 34, 34, 1, 35, 37, 38, 39, 39, 1, 40, 42, 43, 44, 44, 1, 45, 47, 48, 49, 49, 1, 50, 52, 53, 54, 54, 1, 55, 57, 58, 59, 59, 1, 60, 62, 63, 64, 64, 1, 65, 67, 68, 69, 69, 1, 70, 72, 73, 74, 74, 1, 75, 77, 78, 79, 79, 1, 80, 82, 83, 84, 84, …] = ???

Find formula expressions for all “???”.

9) Again: e^2 & lateral

9.1)

$$[0;1,\frac{-1}{2},-2,\frac{+3}{2},+2,\frac{-5}{2},-2,\frac{+7}{2},+2,\frac{-9}{2},-2,\frac{+11}{2},+2,...]=$$

7.389056098930650227230427460575007813180315570551847324087127822522573796079057763384312485079121795… =

[7; 2, 1, **1, 3, 18, 5, 1, 1, 6, 30, 8, 1,** 1, 9, 42, 11, 1, 1, 12, 54, 14, 1, 1, 15, 66, 17, 1, 1, 18, 78, 20, 1, 1, 21, 90, 23, 1, 1, 24, 102, 26, 1, 1, 27, 114, 29, 1, 1, 30, 126, 32, 1, 1, 33, 138, 35, 1, 1, 36, 150, 38, 1, 1, 39, 162, 41, 1, 1, 42, 174, 44, 1, 1, 45, 186, 47, 1, 1, 48, 198, 50, 1, 1, 51, 210, 53, 1, 1, 54, 222, 56, 1, 1, 57, 234, 59, 1, 1, 60, 246, 62, 1, 1, 63, 258, 65, 1, 1, 66, 270, 68, 1, 1, 69, 282, 71, 1, 1, 72, 294, 74, 1, 1, 75, 306, 77, 1, 1, 78, 318, 80, 1, 1, 81, 330, 83, 1, 1, 84, 342, 86, 1, 1, 87, 354, 89, 1, 1, 90, 366, 92, 1, 1, 93, 378, 95, 1, 1, 96, 390, 98, 1, 1, 99, 402, 101, 1, 1, 102, 414, 104, 1, 1, 105, 426, 107, 1, 1, 108, 438, 110, 1, 1, 111, 450, 113, 1, 1, 114, 462, 116, 1, 1, 117, 474, 119, 1, 1, …] $= e^2$

$$[0;1,\frac{-1}{2},-2,\frac{+3}{2},+2]=7$$

9.2)

$$[0;1,\frac{+1}{2},+2,\frac{+3}{2},+2,\frac{+5}{2},+2,\frac{+7}{2},+2,\frac{+9}{2},+2,\frac{+11}{2},+2,...]=$$

0.472854499113084343069904341744489966146404169323917727564183717975723457323448665 87

83734141308844870… =

[0; 2, 8, 1, 2, 2, 3, 1, 20, 1,
5, 2, 6, 1, 32, 1, 8, 2, 9, 1, 44, 1,
11, 2, 12, 1, 56, 1, 14, 2, 15, 1, 68, 1,
17, 2, 18, 1, 80, 1, 20, 2, 21, 1, 92, 1,
23, 2, 24, 1, 104, 1, 26, 2, 27, 1, 116, 1,
29, 2, 30, 1, 128, 1, 32, 2, 33, 1, 140, 1,
35, 2, 36, 1, 152, 1, 38, 2, 39, 1, 164, 1,
41, 2, 42, 1, 176, 1, 44, 2, 45, 1, 188, 1,
47, 2, 48, 1, 200, 1, 50, 2, 51, 1, 212, 1,
53, 2, 54, 1, 224, 1, 56, 2, 57, 1, 236, 1,
59, 2, 60, 1, 248, 1, 62, 2, 63, 1, 260, 1,
65, 2, 66, 1, 272, 1, 68, 2, 69, 1, 284, 1,
71, 2, 72, 1, 296, 1, 74, 2, 75, 1, 308, 1,
77, 2, 78, 1, 320, 1, 80, 2, 81, 1, 332, 1,
83, 2, 84, 1, 344, 1, 86, 2, 87, 1, 356, 1,
89, 2, 90, 1, 368, 1, 92, 2, 93, 1, 380, 1,
95, 2, 96, 1, 392, 1, 98, 2, 99, 1, 404, 1,
101, 2, 102, 1, 416, 1, 104, 2, 105, 1, 428, 1,
107, 2, 108, 1, 440, 1, 110, 2, 111, 1, 452, 1,
113, 2, 114, 1, 464, 1, 116, 2, 117, 1, 476, 1,
119, 2, 120, 1, 488, 1, 122, 2, 123, 1, 500, 1,
125, 2, 126, 1, 512, 1, 128, 2, 129, 1, 524, 1,
131, 2, 132, 1, 536, 1, 134, 2, 135, 1, 548, 1,
137, 2, 138, 1, 560, 1, 140, 2, 141, 1, 572, 1,
143, 2, 144, 1, 584, 1, 146, 2, 147, 1, 596, 1, …]

= ???

9.3)

$$[0;1,\frac{-1}{2},+2,\frac{+3}{2},+2,\frac{-5}{2},+2,\frac{+7}{2},+2,\frac{-9}{2},+2,\frac{+11}{2},+2,...]=$$

-0.10369988471575033818916985088485235731014429174141985756780300515898022542472887712288102507675575623… =

-[0; 9, 1, 1, 1, 4, 14, 6, 1, **1, 5, 34, 7, 1, 1, 10, 38, 12, 1**, 1, 11, 58, 13, 1, 1, 16,
62, 18, 1, 1, 17, 82, 19, 1, 1, 22, 86, 24, 1, 1, 23, 106, 25, 1, 1, 28, 110, 30, 1,
1, 29, 130, 31, 1, 1, 34, 134, 36, 1, 1, 35, 154, 37, 1, 1, 40, 158, 42, 1, 1, 41,
178, 43, 1, 1, 46, 182, 48, 1, 1, 47, 202, 49, 1, 1, 52, 206, 54, 1, 1, 53, 226, 55,
1, 1, 58, 230, 60, 1, 1, 59, 250, 61, 1, 1, 64, 254, 66, 1, 1, 65, 274, 67, 1, 1,
70, 278, 72, 1, 1, 71, 298, 73, 1, 1, 76, 302, 78, 1, 1, 77, 322, 79, 1, 1, 82, 326,
84, 1, 1, 83, 346, 85, 1, 1, 88, 350, 90, 1, 1, 89, 370, 91, 1, 1, 94, 374, 96, 1,
1, 95, 394, 97, 1, 1, 100, 398, 102, 1, 1, 101, 418, 103, 1, 1, 106, 422, 108, 1, 1,
107, 442, 109, 1, 1, 112, 446, 114, 1, 1, 113, 466, 115, 1, 1, 118, 470, 120, 1, 1,
119, 490, 121, 1, 1, 124, 494, 126, 1, 1, 125, 514, 127, 1, 1, 130, 518, 132, 1, 1,
131, 538, 133, 1, 1, 136, 542, 138, 1, 1, 137, 562, 139, 1, 1, 142, 566, 144, 1, 1,
143, 586, 145, 1, 1, 148, 590, 150, 1, 1, 149, 610, 151, 1, 1, 154, 614, 156, 1, 1,
155, 634, 157, 1, 1, 160, 638, 162, 1, 1, 161, 658, 163, 1, 1, 166, 662, 168, 1, 1,

167, 682, 169, 1, 1, 172, 686, 174, 1, 1, 173, 706, 175, 1, 1, …] = ???

9.4)

$$[0;1,\frac{+1}{2},-2,\frac{+3}{2},+2,\frac{+5}{2},-2,\frac{+7}{2},+2,\frac{+9}{2},-2,\frac{+11}{2},+2,...]=$$

-0.21795809846086199811129429910638368281310334692686284848144510024530033383630420460406665458935276320… =

-[0; 4, 1, 1, 2, 2, 1, 16, 1,

```
 4, 2,   5, 1,  28, 1,   7, 2,   8, 1,  40, 1,
10, 2,  11, 1,  52, 1,  13, 2,  14, 1,  64, 1,
16, 2,  17, 1,  76, 1,  19, 2,  20, 1,  88, 1,
22, 2,  23, 1, 100, 1,  25, 2,  26, 1, 112, 1,
28, 2,  29, 1, 124, 1,  31, 2,  32, 1, 136, 1,
34, 2,  35, 1, 148, 1,  37, 2,  38, 1, 160, 1,
40, 2,  41, 1, 172, 1,  43, 2,  44, 1, 184, 1,
46, 2,  47, 1, 196, 1,  49, 2,  50, 1, 208, 1,
52, 2,  53, 1, 220, 1,  55, 2,  56, 1, 232, 1,
58, 2,  59, 1, 244, 1,  61, 2,  62, 1, 256, 1,
64, 2,  65, 1, 268, 1,  67, 2,  68, 1, 280, 1,
70, 2,  71, 1, 292, 1,  73, 2,  74, 1, 304, 1,
76, 2,  77, 1, 316, 1,  79, 2,  80, 1, 328, 1,
82, 2,  83, 1, 340, 1,  85, 2,  86, 1, 352, 1,
88, 2,  89, 1, 364, 1,  91, 2,  92, 1, 376, 1,
94, 2,  95, 1, 388, 1,  97, 2,  98, 1, 400, 1,
100, 2, 101, 1, 412, 1, 103, 2, 104, 1, 424, 1,
106, 2, 107, 1, 436, 1, 109, 2, 110, 1, 448, 1,
112, 2, 113, 1, 460, 1, 115, 2, 116, 1, 472, 1,
118, 2, 119, 1, 484, 1, 121, 2, 122, 1, 496, 1,
124, 2, 125, 1, 508, 1, 127, 2, 128, 1, 520, 1,
130, 2, 131, 1, 532, 1, 133, 2, 134, 1, 544, 1,
136, 2, 137, 1, 556, 1, 139, 2, 140, 1, 568, 1,

142, 2, 143, 1, 580, 1, 145, 2, 146, 1, 592, 1, ...]
```

$= ???$

9.5)

$$[0;1,\frac{+1}{2},+2,\frac{-3}{2},-2,\frac{+5}{2},+2,\frac{-7}{2},-2,\frac{+9}{2},+2,\frac{-11}{2},-2,...] =$$

```
0.536289441747876912526090013366862735010157831874120378391364079957153879290634921305691714998568006 0…
```

```
[0; 1, 1, 6, 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, 8, 1, 1, 9, 42, 11, 1, 1, 12, 54, 14, 1, 1,
15, 66, 17, 1, 1, 18, 78, 20, 1, 1, 21, 90, 23, 1, 1, 24, 102, 26, 1, 1, 27, 114, 29,
1, 1, 30, 126, 32, 1, 1, 33, 138, 35, 1, 1, 36, 150, 38, 1, 1, 39, 162, 41, 1, 1,
42, 174, 44, 1, 1, 45, 186, 47, 1, 1, 48, 198, 50, 1, 1, 51, 210, 53, 1, 1, 54, 222,
56, 1, 1, 57, 234, 59, 1, 1, 60, 246, 62, 1, 1, 63, 258, 65, 1, 1, 66, 270, 68, 1,
1, 69, 282, 71, 1, 1, 72, 294, 74, 1, 1, 75, 306, 77, 1, 1, 78, 318, 80, 1, 1, 81,
330, 83, 1, 1, 84, 342, 86, 1, 1, 87, 354, 89, 1, 1, 90, 366, 92, 1, 1, 93, 378, 95,
1, 1, 96, 390, 98, 1, 1, 99, 402, 101, 1, 1, 102, 414, 104, 1, 1, 105, 426, 107, 1,
1, 108, 438, 110, 1, 1, 111, 450, 113, 1, 1, 114, 462, 116, 1, 1, 117, 474, 119, 1,
1, 120, 486, 122, 1, 1, 123, 498, 125, 1, 1, 126, 510, 128, 1, 1, 129, 522, 131, 1,
1, 132, 534, 134, 1, 1, 135, 546, 137, 1, 1, 138, 558, 140, 1, 1, 141, 570, 143, 1,
1, 144, 582, 146, 1, 1, 147, 594, 149, 1, 1, 150, 606, 152, 1, 1, 153, 618, 155, 1,
1, 156, 630, 158, 1, 1, 159, 642, 161, 1, 1, 162, 654, 164, 1, 1, 165, 666, 167, 1,

1, 168, 678, 170, 1, 1, 171, 690, 173, 1, 1, 174, 702, 176, 1, 1, 177, 714, 179,…]
```

$=$

$$\frac{e^2}{2e^2-1}$$

9.1), 9.3), 9.5)
9.2), 9.4)
} are strukturally similar.

Find formula expressions for 9.2), 9.3), and 9.4).

10)
Calculate and discuss the graph of the scf-defined function (← 9.2))
f(x) =

$$[0;1,\frac{+1}{x},+2,\frac{+3}{x},+2,\frac{+5}{x},+2,\frac{+7}{x},+2,\frac{+9}{x},+2,\frac{+11}{x},+2,...]$$

and of its first derivative.
Find the power series representation of f(x).
More (← 9)).