Unit n°117 continuation 3

Bessel versus Tangent functions

4) Continued fraction representations of e^2

$$e^2 = [7;2,1,1,3,18,5,1,1,6,30,8,1,1,9,42,11,\ldots]$$

$$= [7,\frac{5}{2},14,\frac{9}{2},22,\frac{13}{2},30,\ldots]$$

$$= [0;1,\frac{-1}{2},-2,\frac{+3}{2},+2,\frac{-5}{2},-2,\frac{+7}{2},+2,\frac{-9}{2},-2,\frac{+11}{2},+2,\ldots]$$

Lateral: how about

$$[0;1,\frac{+1}{2},+2,\frac{+3}{2},+2,\frac{+5}{2},+2,\frac{+7}{2},+2,\frac{+9}{2},+2,\frac{+11}{2},+2,\ldots]$$

$$[0;1,\frac{-1}{2},+2,\frac{+3}{2},+2,\frac{-5}{2},+2,\frac{+7}{2},+2,\frac{-9}{2},+2,\frac{+11}{2},+2,\ldots]$$

$$[0;1,\frac{+1}{2},-2,\frac{+3}{2},+2,\frac{+5}{2},-2,\frac{+7}{2},+2,\frac{+9}{2},-2,\frac{+11}{2},+2,\ldots]$$

$$[0;1,\frac{+1}{2},+2,\frac{-3}{2},-2,\frac{+5}{2},+2,\frac{-7}{2},-2,\frac{+9}{2},+2,\frac{-11}{2},-2,\ldots]$$

????

Solution

$$[0;1,\frac{+1}{2},+2,\frac{-3}{2},-2,\frac{+5}{2},+2,\frac{-7}{2},-2,\frac{+9}{2},+2,\frac{-11}{2},-2,\ldots] =$$

$$\frac{e^2}{2e^2-1}$$

More is your turn.



Photo: Jonathan Stoschek

5) First derivative of the tangent functions

$$\left.\begin{matrix}\tan x\\ \tanh x\end{matrix}\right\} = \cfrac{1}{\cfrac{1}{x} \mp \cfrac{1}{\cfrac{3}{x} \mp \cfrac{1}{\cfrac{5}{x} \mp \cfrac{1}{\cfrac{7}{x} \mp \cfrac{1}{\cfrac{9}{x} \mp \cfrac{1}{\cfrac{11}{x} \mp \ldots}}}}}} =$$

$$[0;\frac{1}{x},\mp\frac{3}{x},\frac{5}{x},\mp\frac{7}{x},\frac{9}{x},\mp\frac{11}{x},\ldots] =$$

$$[c_0;\frac{c_1}{x},\frac{c_2}{x},\frac{c_3}{x},\frac{c_4}{x},\frac{c_5}{x},\frac{c_6}{x},\ldots] =$$

$$\begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \cdots \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \cdots \\ \pm x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \cdots \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdots \\ \pm x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \cdots \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \cdots \\ \pm x^{11} & 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix} =$$

$$x^1 \pm \frac{1}{3}x^3 + \frac{2}{15}x^5 \pm \frac{17}{315}x^7 + \frac{62}{2835}x^9 \pm \frac{50521}{3628800}x^{11} + \ldots$$

$$\left.\begin{aligned} \frac{\mathrm{dtanx}}{\mathrm{dx}} &= \frac{1}{\cos^2 x} \\ \frac{\mathrm{dtanhx}}{\mathrm{dx}} &= \frac{1}{\cosh^2 x} \end{aligned}\right\} = \mathrm{gcf}(x) =$$

$$[a_0;\frac{a_1}{x},\frac{a_2}{x},\frac{a_3}{x},\frac{a_4}{x},\frac{a_5}{x},\frac{a_6}{x},...]??? =$$

$$\begin{vmatrix}
x^0 & \frac{2}{2!} & \frac{8}{4!} & \frac{32}{6!} & \frac{128}{8!} & \frac{512}{10!} & \frac{2048}{12!} & \cdots \\
-x^2 & \frac{1}{0!} & \frac{2}{2!} & \frac{8}{4!} & \frac{32}{6!} & \frac{128}{8!} & \frac{512}{10!} & \cdots \\
\pm x^4 & 0 & \frac{1}{0!} & \frac{2}{2!} & \frac{8}{4!} & \frac{32}{6!} & \frac{128}{8!} & \cdots \\
-x^6 & 0 & 0 & \frac{1}{0!} & \frac{2}{2!} & \frac{8}{4!} & \frac{32}{6!} & \cdots \\
\pm x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2}{2!} & \frac{8}{4!} & \cdots \\
-x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2}{2!} & \cdots \\
\pm x^{11} & 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{vmatrix} =$$

$$1 \pm x^2 + \frac{2}{3}x^4 \pm \frac{17}{45}x^6 + \frac{62}{315}x^8 \pm \frac{1382}{14175}x^{10} + ...$$

Open Problems, waiting for you

Find – based on 3)

– catch as catch can

the scf coefficients a_i.

Find a direct way from

$$f(x) = [c_0; \frac{c_1}{x}, \frac{c_2}{x}, \frac{c_3}{x}, \frac{c_4}{x}, \frac{c_5}{x}, ...]$$

to

$$f'(x) = [a_0; \frac{a_1}{x}, \frac{a_2}{x}, \frac{a_3}{x}, \frac{a_4}{x}, \frac{a_5}{x}, ...]$$

$$= \frac{d[c_0; \frac{c_1}{x}, \frac{c_2}{x}, \frac{c_3}{x}, \frac{c_4}{x}, \frac{c_5}{x}, ...]}{dx}$$

for

$$** \; f(x) = \begin{cases} \tan x \\ \tanh x \end{cases} = [0; \frac{1}{x}, \mp\frac{3}{x}, \frac{5}{x}, \mp\frac{7}{x}, \frac{9}{x}, \mp\frac{11}{x}, ...]$$

$$** \; f(x) = \begin{cases} \sin x \\ \sinh x \end{cases} = [0; \frac{1}{x}, ...]$$

$$** \text{ all } f(x) = [c_0; \frac{c_1}{x}, \frac{c_2}{x}, \frac{c_3}{x}, \frac{c_4}{x}, \frac{c_5}{x}, ...]$$

→ **scf differentiation rules**.