# Unit n°117 continuation 2

## Bessel versus Tangent functions

1) Bessel functions [2,3,4]

$$\left.\begin{matrix} J_n(x) \\ I_n(x) \end{matrix}\right\} = \sum_{i=0}^{\infty} (\mp 1)^i \frac{\left(\frac{x}{2}\right)^{2i+n}}{i!(i+n)!}$$

$$\left.\begin{matrix} \frac{J_1(x)}{J_0(x)} \\ \frac{I_1(x)}{I_0(x)} \end{matrix}\right\} = \cfrac{1}{\frac{2}{x} \mp \cfrac{1}{\frac{4}{x} \mp \cfrac{1}{\frac{6}{x} \mp \cfrac{1}{\frac{8}{x} \mp \cfrac{1}{\frac{10}{x} \mp \cfrac{1}{\frac{12}{x} \mp \ldots}}}}}} =$$

$$[0; \frac{2}{x}, \mp\frac{4}{x}, \frac{6}{x}, \mp\frac{8}{x}, \frac{10}{x}, \mp\frac{12}{x}, \ldots] =$$

$$\begin{vmatrix} 0 & \frac{1}{0!1!} & \frac{1}{1!2!} & \frac{1}{2!3!} & \frac{1}{3!4!} & \frac{1}{4!5!} & \frac{1}{5!6!} & \cdots \\ -\left(\frac{x}{2}\right)^1 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \frac{1}{4!4!} & \frac{1}{5!5!} & \cdots \\ \pm\left(\frac{x}{2}\right)^3 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \frac{1}{4!4!} & \cdots \\ -\left(\frac{x}{2}\right)^5 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \cdots \\ \pm\left(\frac{x}{2}\right)^7 & 0 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \cdots \\ -\left(\frac{x}{2}\right)^9 & 0 & 0 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \cdots \\ \pm\left(\frac{x}{2}\right)^{11} & 0 & 0 & 0 & 0 & 0 & \frac{1}{0!0!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix} =$$

$$\left(\frac{x}{2}\right)^1 \pm \frac{1}{2}\left(\frac{x}{2}\right)^3 + \frac{1}{3}\left(\frac{x}{2}\right)^5 \pm \frac{11}{48}\left(\frac{x}{2}\right)^7 + \frac{19}{120}\left(\frac{x}{2}\right)^9$$

$$\pm \frac{473}{4320}\left(\frac{x}{2}\right)^{11} + \frac{229}{3024}\left(\frac{x}{2}\right)^{13} \pm \ldots$$

$$\frac{J_1(2)}{J_0(2)} = 2.575920322...$$

$$= [2;1,1,2,1,3,1,4,1,5,1,6,1,...]$$

$$= [0;1,-2,3,-4,5,-6,...]$$

$$\frac{I_1(2)}{I_0(2)} = 0.697774658...$$

$$= [0;1,2,3,4,5,6,...]$$

$$= 2[0;2,1,6,2,10,3,14,4,18,5,22,6,\ldots]$$

$$\frac{e-1}{e+1} = 0.462117157...$$

$$= [0;2,6,10,14,18,22,\ldots]$$

Entanglement of I and e

2) Tangent functions [6,7]

$$\left.\begin{matrix}\tan\ x\\ \tanh x\end{matrix}\right\}=\cfrac{1}{\cfrac{1}{x}\mp\cfrac{1}{\cfrac{3}{x}\mp\cfrac{1}{\cfrac{5}{x}\mp\cfrac{1}{\cfrac{7}{x}\mp\cfrac{1}{\cfrac{9}{x}\mp\cfrac{1}{\cfrac{11}{x}\mp\ldots}}}}}}=$$

$$[0;\frac{1}{x},\mp\frac{3}{x},\frac{5}{x},\mp\frac{7}{x},\frac{9}{x},\mp\frac{11}{x},\ldots]=$$

$$\begin{vmatrix}
0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \cdots \\
-x^{1} & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \cdots \\
\pm x^{3} & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \cdots \\
-x^{5} & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdots \\
\pm x^{7} & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \cdots \\
-x^{9} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \cdots \\
\pm x^{11} & 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{vmatrix}=$$

$$x^1 \pm \frac{1}{3}x^3 + \frac{2}{15}x^5 \pm \frac{17}{315}x^7 + \frac{62}{2835}x^9 \pm \frac{50521}{3628800}x^{11} + \ldots$$

tanh 1 = [0;1,3,5,7,9,11,13,...]
tan 1 = [1;1,1,3,1,5,1,7,1,9,1,11,…]
= [0;1,-3,5,-7,9,-11,13,…]

3) For comparison: exponential function $e^x$ and tangent function tan x

$$e^x = \cfrac{1}{1 - \cfrac{1}{\frac{1}{x} + \cfrac{1}{2 - \cfrac{1}{\frac{3}{x} + \cfrac{1}{2 - \cfrac{1}{\frac{5}{x} + \cfrac{1}{2 - \cfrac{1}{\frac{7}{x} + \cfrac{1}{2 - \ldots}}}}}}}}} =$$

$$[0;1,-\frac{1}{x},-2,\frac{3}{x},2,-\frac{5}{x},-2,\frac{7}{x},2,\ldots] =$$

$$\begin{vmatrix} x^0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \cdots \\ -x^1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \cdots \\ x^2 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots \\ -x^3 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \cdots \\ x^4 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \cdots \\ -x^5 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \cdots \\ x^6 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix} =$$

$$\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$$

$$e = [2;1,2,1,1,4,1,1,6,1,1,8,\ldots]$$

$$= \frac{1}{2}[5;2,3,2,3,1,2,1,3,4,3,1,4,1,3,6,3,1,6,1,\ldots]$$

$$= 3 - 2[0;7,10,14,18,22,26,30,\ldots]$$

$$= [3, -\frac{7}{2}, -20, -7, -36, -11, -52, -15\ldots]$$

## References


[1] Weisstein, Eric W. "Bessel Function of the First Kind." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

[2] https://en.wikipedia.org/wiki/Bessel_function

[3] www.aip.de/groups/soe/local/numres/bookcpdf/c6-7.pdf