Unit n°103

e^{π} algorithmics, continuation 1

The decimal representation of

e^{π} = 23.140692632779269005729086367948547380266106242600211993445046409524342350690452783516971997067549229…

(Gelfond's constant [1,2,3])

can also be computed efficiently using the trigonometric function based algorithm consisting of

** the recursion equation

$$\sin\alpha(t+1) = \tan^2\frac{\alpha(t)}{2} \quad ; t=0,1,2,3,... \qquad (1)$$

with the initial value

$$\alpha(0) = \frac{\pi}{4} \qquad (2)$$

** and the downstream output transformation

$$y(t) = \sqrt[2^{(t-1)}]{\frac{2}{\tan\frac{\alpha(t)}{2}}} \qquad (3)$$

$$\underline{\underline{e^{\pi} = \lim_{t\to\infty} y(t)}} \qquad (4)$$

The execution records for α(t) and y(t)

$$\alpha(0) = \frac{\pi}{4}$$
$$= 0.785398163397448309615660$$
$$845820...$$

$$\alpha(1) = \frac{\pi}{18.21994774...}$$
$$= 0.172425997712849066185386$$
$$276946...$$

$$\alpha(2) = \frac{\pi}{420.5761267...}$$
$$= 0.007469736194076241987469$$
$$42156742...$$

$$\alpha(3) = \frac{\pi}{225213.9636...}$$
$$= 0.000013949369424609787237 3$$
$$426833...$$

$\alpha(t+1) \approx \left(\frac{\alpha(t)}{2}\right)$ the larger t, the more accurate.

y(3) = 23.140692632779269005729086 3

6794854738026610 6valid
y(4) = 23.14069263277926900572908 63
6794854738026610624260021 19
9344504640952434235069045 27
8351697valid

show

1* per recursion step in equ. (1) we obtain a doubling of valid digits in $y(t) \rightarrow e^{\pi}$,

2* the present algorithm equs (1,2,3) has – despite its very fast convergence – a serious disadvantage: its huge intrinsic numerical disbalance, y(t) increases exponentially and equ. (3) reduces these values exponentially to 23.14…

Problems

1* Discuss the output $y(\infty)$ of the algorithm equs (1,3) for various initial values $\alpha(0)$:

$$\alpha(0) = \frac{\pi}{2} \rightarrow \quad y(\infty) = e^{0} = 1$$

$$\alpha(0) = \frac{\pi}{3} \rightarrow \quad y(\infty) = 11.655591214722821336137680$$

$$295585901...$$

$$\alpha(0) = \frac{\pi}{4} \rightarrow \quad y(\infty) = e^{\pi}$$

$$= 23.140692632779269005729086$$

$$367948547380266106242600211$$

$$993445046409524342350690...$$

$$\alpha(0) = \frac{\pi}{5} \rightarrow \quad y(\infty) = 37.782638110455505307127440 3$$

$$541115483959706262251029...$$

$$\alpha(0) = \frac{\pi}{6} \rightarrow \quad y(\infty) = 55.640911777665069772675384 7$$

$$570667785691378515266904 62...$$

2* Discuss the function $y(\infty) = f(\alpha(0))$.

3* Modify the algorithm equs (1,2,3) so that the ride on the big dipper 2* is significantly reduced or completely eliminated.

4* Read Unit 1, p. 12
Convergence acceleration via algorithmic model „Engine + Load“.
Engine = fast converging dynamic input follower, load = slow converging numerical

algorithm.
Try to distill from this very fast e^{π} algorithm
** by a suitable normalization
** a high speed dynamic input follower engine.

5* Find a e^{π}-similar algorithm for $\sqrt[\pi]{e}$.

6* Compare the e^{π} algorithms Unit 101 and Unit 103.

References

[1] Weisstein, Eric W. "Gelfond's Constant." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/GelfondsConstant.html

[2] https://en.wikipedia.org/wiki/Gelfond's_constant

[3] https://oeis.org/A039661 , http://oeis.org/A058287

Links to Deb 16
Unit 60
Unit 101