Unit n° 100

Real polynomials and their properties, determinant representation

Start

$$p_n(x) = \sum_{i=0}^{n} a_i x^i;\ a_0 = 1$$

$$= \begin{vmatrix} 1 & -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_n \\ x^1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ x^2 & 0 & 1 & 0 & 0 & \cdots & 0 \\ x^3 & 0 & 0 & 1 & 0 & \cdots & 0 \\ x^4 & 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ x^n & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\frac{1}{p_n(x)}=\frac{1}{\sum_{i=0}^{n}a_i x^i};\ a_0=1$$

$$=\begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ x^1 & 1 & a_1 & a_2 & a_3 & \ddots & \vdots \\ x^2 & 0 & 1 & a_1 & a_2 & \ddots & a_4 \\ x^3 & 0 & 0 & 1 & a_1 & \ddots & a_3 \\ x^4 & 0 & 0 & 0 & 1 & \ddots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & a_1 \\ x^n & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\frac{\sum_{i=0}^{\infty}b_i x^i}{\sum_{i=0}^{\infty}a_i x^i}=-\begin{vmatrix} 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ x^0 & 1 & a_1 & a_2 & a_3 & a_4 & \cdots \\ x^1 & 0 & 1 & a_1 & a_2 & a_3 & \cdots \\ x^2 & 0 & 0 & 1 & a_1 & a_2 & \cdots \\ x^3 & 0 & 0 & 0 & 1 & a_1 & \cdots \\ x^4 & 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix};\ a_0=1$$

$$\frac{\sum_{i=0}^{\infty} b_{2i+1} x^{2i+1}}{\sum_{i=0}^{\infty} a_{2i} x^{2i}} = -\begin{vmatrix} 0 & b_1 & b_3 & b_5 & b_7 & b_9 & \cdots \\ x^1 & 1 & a_2 & a_4 & a_6 & a_8 & \cdots \\ x^3 & 0 & 1 & a_2 & a_4 & a_6 & \cdots \\ x^5 & 0 & 0 & 1 & a_2 & a_4 & \cdots \\ x^7 & 0 & 0 & 0 & 1 & a_2 & \cdots \\ x^9 & 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix} ; a_0 = 1$$

Link to Unit 4 and 64

Change the lobster:

$$p_n(x) = \sum_{i=0}^{n} A_i x^{n-i} = A_0 \prod_{j=1}^{n} (x - x_j)$$

Assigned basis determinant

$$\underline{D} = \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & \cdots \\ A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & \cdots \\ 0 & A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & 0 & 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix}$$

First modification
Delete in $\underline{D}$ the columns 2, 4, 6, … and push the remainder to the new determinant

$$\underline{\Delta}_1 \quad \underline{\Delta}_2 \quad \underline{\Delta}_3 \quad \underline{\Delta}_4 \quad \underline{\Delta}_5 \quad \underline{\Delta}_6$$
$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\underline{\Delta} = \begin{vmatrix} A_1 & A_3 & A_5 & A_7 & A_9 & A_{11} & \cdots \\ A_0 & A_2 & A_4 & A_6 & A_8 & A_{10} & \cdots \\ 0 & A_1 & A_3 & A_5 & A_7 & A_9 & \cdots \\ 0 & A_0 & A_2 & A_4 & A_6 & A_8 & \cdots \\ 0 & 0 & A_1 & A_3 & A_5 & A_7 & \ddots \\ 0 & 0 & A_0 & A_2 & A_4 & A_6 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix}$$

$\underline{\Delta}$ contains information about the position of the n zeros s_j of $p_n(x)$ in the Gauss plane:
Iff all elements of the sequence

$$\underline{\Delta}_0=A_0,\ \underline{\Delta}_1=A_1,\ \underline{\Delta}_2=\begin{vmatrix} A_1 & A_3 \\ A_0 & A_2 \end{vmatrix},$$

$$\underline{\Delta}_3=\begin{vmatrix} A_1 & A_3 & A_5 \\ A_0 & A_2 & A_4 \\ 0 & A_1 & A_3 \end{vmatrix},\ \underline{\Delta}_4=\begin{vmatrix} A_1 & A_3 & A_5 & A_7 \\ A_0 & A_2 & A_4 & A_6 \\ 0 & A_1 & A_3 & A_5 \\ 0 & A_0 & A_2 & A_4 \end{vmatrix},\ldots,$$

$\underline{\Delta}_n=A_n*\underline{\Delta}_{n-1}$
are positive
then all s_j have negative real parts, e.g. are located in the left half-space of the Gauss plane,
$(\forall j)\ 180°<\arg(s_j)<360°$
(Routh-Hurwitz Theorem [1,2]).
Example
$p_3(x) = x^3+12x^2+54x+68$
$A_0=1,\ A_1=12,\ A_2=54,\ A_3=68$
$x_1=\mathbf{-2},\ x_{2,3}=\mathbf{-5}\pm 3j$

$$\Delta = \begin{vmatrix} 12 & 68 & 0 & 0 & \cdots \\ 1 & 54 & 0 & 0 & \cdots \\ 0 & 12 & 68 & 0 & \cdots \\ 0 & 1 & 54 & 0 & \cdots \\ \vdots & & & & \end{vmatrix} \rightarrow$$

$\Delta_0 = 1,\ \Delta_1 = 12,\ \Delta_2 = 580,\ \Delta_3 = 68 * 580$

$(\forall i)\Delta_i$ positive $\Rightarrow$ $(\forall j)\mathrm{Re}(s_j)$ negative

Second and third modification

Find analogues to the first modification for the segments

** $(\forall j)\ 225° < \arg(s_j) < 315°$

** $(\forall j)\ 240° < \arg(s_j) < 300°$.

References

[1] Weisstein, Eric W. "Routh-Hurwitz Theorem." From *MathWorld* --A Wolfram Web Resource. http://mathworld.wolfram.com/Routh-HurwitzTheorem.html

[2] https://de.wikipedia.org/wiki/Hurwitzpolynom

To be continued.

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