Unit n°66

Extension of Unit 5, part **5.1 Eadem mutata resurgo**

Problem corresponding to Unit 1, Babylonian sqr(2) problem
Find a simple, fast, and robust algorithm for the numeric calculation of

$x = sqr(2) + sqr(3) = 3.1462643\ldots$. (1)

Solution

$x = sqr(2) + sqr(3) \rightarrow x^2 = 2 + 2sqr(6) + 3 \rightarrow (x^2 - 5)^2 = 24 \rightarrow$
$x^4 - 10x^2 + 1 = 0 \rightarrow x = (10x^2 - 1)/x^3$

→ "Babylon" recursion approach:

$$x := \frac{1}{2}\left(x + \frac{10x^2 - 1}{x^3}\right) \rightarrow \quad (2)$$

Execution record for the initial value x(0)=5

t	x(t)
0	5.00000000000000000…
1	3.49600000000000000…
2	3.16650406717143267 87…
3	3.14653231943270651 71…
4	3.14626711523199011 34…
5	3.14626439795918981 29…
6	3.14626437022789220 89…
7	3.14626436994489019 58…
8	3.14626436994200211 94…
9	3.14626436994197264 62…
10	3.14626436994197234 54…
11	3.14626436994197234 23…
12	3.14626436994197234 23… stop, fast convergence.

Associated **difference quotient**

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)} = 97.989794855663561963 94...$$

$$= \lim_{t\to\infty} (x(t))^4 \quad !!!$$

$$= (sqr(2) + sqr(3))^4 \qquad (3.1)$$

Your **homework**

Check $\lim_{t\to\infty} d^{(k)}x(t)$ with

$$d^{(k)}x(t) = \frac{d^{(k-1)}x(t) - d^{(k-1)}x(t+1)}{d^{(k-1)}x(t+1) - d^{(k-1)}x(t+2)} \; ; \qquad (3.2)$$

$k=2,3,4,5,...$

$$d^{(0)}x(t) = x(t)$$

$$d^{(1)}x(t) = dx(t)$$

Diskussion!

Comment 1

The foregoing problem is an example for a converging recursion equation

$$x := \varphi(x) \qquad (4)$$

with

$$\lim_{t\to\infty} x(t) = a \qquad (5)$$

and

$$\lim_{t\to\infty} dx(t) = \frac{x(t) - x(t+1)}{x(t+1) - x(t+2)} = a^n \; ; \; n \in \mathbb{N} \qquad (6)$$

** Find more examples for the function $\varphi(x)$ that fulfils in the recursion equation (4) the conditions (5) and(6).

** Find a general solution for this problem.

** Are there possible applications in science and engineering?

Comment 2

How about

$y = sqr(3) - sqr(2) = 0.317837245\ldots$? (7)

Extension 1

Problem

Find a simple and fast algorithm for the numeric calculation of

$x = sqr(a) + sqr(b)\ ;\ a > b$ (8)

Solution

$x = sqr(a) + sqr(b) \rightarrow x^2 = a + 2sqr(ab) + b \rightarrow$

$(x^2 - a - b)^2 = 4ab \rightarrow$

$f(x) = x^4 - 2(a + b)x^2 + (a - b)^2 = 0 \rightarrow$

$x = [2(a + b)x^2 + (a - b)^2]/x^3 \rightarrow$

"Babylon" recursion approach:

$$x := \frac{1}{2}\left(x + \frac{2(a+b)x^2-(a-b)^2}{x^3}\right) = x - \frac{f(x)}{2x^3} \quad (9)$$

Example 1 a=1936; b=77→ sqr(a)+sqr(b)=52.77496439…

Initial value x(0)=45

Execution record

t	x(t)
0	45.0
1	48.27102332…
2	50.47478792…
3	51.68159927…
4	52.27323611…
5	52.54844791…
6	52.67344843…
7	52.72961870…
8	52.75473895…
9	52.76594921...
10	52.77094719…
11	52.77317455…
12	52.77416698…
13	52.77460913…
14	52.77480613…
15	52.77489388…
16	52.77493297..

$a \gg b$ - slowly converging, $dx(t) \approx 9/4$

Modificated "Babylon" recursion approach:

$$x := \frac{2(a+b)x^2-(a-b)^2}{x^3} = \frac{x^4-f((x)}{x^3} = x-\frac{f((x)}{x^3} \qquad (10)$$

Example 2 a=1936; b=77→ sqr(a)+sqr(b)=52.77496438…

Initial value x(0)=45

Execution record

t	x(t)
0	45.000000000000000000…
1	51.542046639231824417 00…
2	52.871888075739108856 87…
3	52.764182041674974263 93…
4	52.776136973671945627 97…
5	52.774836541766201740 20…
6	52.774978322375214738 98…
7	52.774962868453735715 09…
8	52.774964552958610232 65…
9	52.774964369345128116 93…
10	52.774964389359271217 40…
11	52.774964387177699878 26…
12	52.774964387415494396 97…
13	52.774964387389574440 98…
14	52.774964387392399754 71…
15	52.774964387392091791 31…
16	52.774964387392125359 78…
17	52.774964387392121700 76…
18	52.774964387392122099 60…
19	52.774964387392122056 13…
20	52.774964387392122060 87…
21	52.774964387392122060 35…
22	52.774964387392122060 41…
23	52.774964387392122060 40…
24	52.774964387392122060 40…stop

Better converging than equ. (9), oscillating.

Associated **difference quotient**

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)} = -9.174186822477056686 52 .$$

$$= ???$$

Extension 2

Problem

Find a simple and fast algorithm for the numeric calculation of

$x = sqr(n) + sqr(n+1)\ ;\ n \in \mathbb{N}$ (11)

Solution

a=n, b=n+1 →

$$x := \frac{1}{2}\left(x + \frac{2(2n+1)x^2-1}{x^3}\right) \quad (12)$$

$$y = \sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} + \sqrt{n}\right)^{-1} \quad (14)$$

Example 3 n=17

x = sqr(17) + sqr(18) = 8.365746313…

$$x := \frac{1}{2}\left(x + \frac{70x^2-1}{x^3}\right) \quad (15)$$

Initial value x(0)=10

Execution record

t	x(t)
0	10.000000000000000000…
1	8.499500000000000000…
2	8.366824977414759174 23…
3	8.365746602423121025 52…
4	8.365746312796094477 07…
5	8.365746312736957772 33…
6	8.365746312736945698 69…
7	8.365746312736945696 22…
8	8.365746312736945696 22…stop, fast convergence

Associated **difference quotient**

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)} = 4897.999795835026197767 14...$$

.

$$= \lim_{t\to\infty} (x(t))^4 \quad !!!$$

$$= (\text{sqr}(17) + \text{sqr}(18))^4 \qquad (16)$$

Example 4 n=17777777

x = sqr(17777777) + sqr(17777778) = 8432.740361…

$$\approx \sqrt{71111110} \quad !!!$$

$$x := \frac{1}{2}\left(x + \frac{71111110x^2 - 1}{x^3}\right) \qquad (17)$$

Initial value x(0)=10

Execution record

t	x(t)
0	10.000000000000000000…
1	3555560.499500000000000000…
2	1777790.24973593892580
3	888915.124727348420111552 88…
4	444497.561182471342955431 03…
5	222328.771030115501545425 32…
6	111324.308834182219054083 41…
7	55981.541578550697570256 05…
8	28625.900762431055785466 65…
9	15555.026754465011708684 66…
10	10063.305331257522885907 00…
11	8564.841198581376289461 10…
12	8433.759097649441183632 95…
13	8432.740422762846464567 64…
14	8432.740361234892765135 72…
15	8432.740361234892540671 96…
16	8432.740361234892540671 96…stop

Strong attractor: ** initially halving
** then fast convergence,
** independent of initial value.

Associated **difference quotient**

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)}$$

$$=5056789965432097.99999999999999980224\,.$$

$$=\lim_{t\to\infty} (x(t))^4 \;!!!$$

$$= (\mathrm{sqr}(17777777) + \mathrm{sqr}(17777778))^4 \qquad (18)$$

Lateral finds

With the abbreviated form

$$a\underbrace{bbbbbbb...b}_{n\ \text{times}} = ab^{\underline{n}} \qquad (19.1)$$

we find

$$** \quad x(n) = \frac{71^{\underline{n}}}{17^{\underline{n}}}\ ,\ \lim_{n\to\infty} x(n) = 4\ ,\ \lim_{n\to\infty} dx(n) = 10 \qquad (19.2)$$

$$** \quad 71^{\underline{2n}} = (79^{\underline{n}}) * (8^{\underline{n-1}}9) \qquad (19.3)$$

Extension 3

Problem

Find a simple and fast algorithm for the numeric calculation of

$$x(m,n) = \sum_{i=m}^{n} \sqrt{i}\ ;\ m,n \in \mathbb{N} \qquad (20)$$

Extension 4

Problem

Find a simple and fast algorithm for the numeric calculation of

$$x = \sqrt[3]{n} + \sqrt[3]{n+1}\ ;\ n \in \mathbb{N} \qquad (21)$$

Extension 5

Problem

Find a simple and fast algorithm for the numeric calculation of

$$x = \sqrt{n} + \sqrt{n+2}\ ;\ n \in \mathbb{N} \qquad (22.1)$$

Recurrence equation

$$x := \frac{1}{2}\left(x + 4\frac{(n+1)x^2 - 1}{x^3}\right) \qquad (22.2)$$

Example n = 3 → x =sqr(3)+sqr(5)=3.968118785…

Initial value x(0)=1
Execution record

t	x(t)
0	1.000000000000000000…
1	6.500000000000000000…
2	4.473486572598998634 50…
3	4.002717151627099954 09…
4	3.968814519058256464 99…
5	3.968130065568106311 58…
6	3.968118967074303366 20…
7	3.968118788005009682 21…
8	3.968118785116039683 94…
9	3.968118785069431264 52…
10	3.968118785068679320 15…
11	3.968118785068667188 86…
12	3.968118785068666993 14…
13	3.968118785068666989 98…
14	3.968118785068666989 93…
15	3.968118785068666989 93…stop

Associated **difference quotient**

$$\lim_{t\to\infty} dx(t) = \frac{x(t)-x(t+1)}{x(t+1)-x(t+2)} = 61.983866769659335081 43...$$

$$= \lim_{t\to\infty} (x(t))^{?} \; ?$$

(22.3)

Discussion!

References

http://www.mathpages.com/home/kmath111/kmath111.htm