Unit 64

Comparative determinant representation of the polynomial $p_n(x)=\sum_{i=0}^{n} a_i x^i$ and of its operations

64.1 Determinant representation of the polynomial $p_n(x)=1+\sum_{i=1}^{n} a_i x^i$

$$1+\sum_{i=1}^{n} a_i x^i =$$

$$\begin{vmatrix}
1 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \dots & -a_n \\
x^1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\
x^2 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\
x^3 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\
x^4 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\
x^5 & 0 & 0 & 0 & 0 & 1 & & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \\
x^n & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{vmatrix} \tag{1.1}$$

$$1+\sum_{i=0}^{n} a_i x^i = \begin{vmatrix} 1 & -a_1 & -a_2 & -a_3 & \dots & -a_n \\ x & 1 & 0 & 0 & & 0 \\ 0 & x & 1 & 0 & & 0 \\ 0 & 0 & x & 1 & & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & 0 & x & 1 \end{vmatrix} \tag{1.2}$$

64.2 Determinant representation of $\left(1+\sum_{i=1}^{n} a_i x^i\right)^{-1}$

$$\left(1+\sum_{i=1}^{n} a_i x^i\right)^{-1} = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \rightarrow \\ x^1 & 1 & a_1 & a_2 & a_3 & a_4 & a_5 & \\ x^2 & 0 & 1 & a_1 & a_2 & a_3 & a_4 & \\ x^3 & 0 & 0 & 1 & a_1 & a_2 & a_3 & \\ x^4 & 0 & 0 & 0 & 1 & a_1 & a_2 & \\ x^5 & 0 & 0 & 0 & 0 & 1 & a_1 & \\ x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & & \searrow \end{vmatrix} \tag{2}$$

64.3 Determinant representation of

$$\left(1+\sum_{i=1}^{n} a_i x^i\right)^k$$

$$\left(1+\sum_{i=1}^{n} a_i x^i\right)^k =$$

$$\begin{vmatrix}
1 & -k\,a_1 & -k\,a_2 & -k\,a_3 & -k\,a_4 & -k\,a_5 & -k\,a_6 & \rightarrow \\
x^1 & 1 & \frac{1-k}{2}a_1 & \frac{1-2k}{3}a_2 & \frac{1-3k}{4}a_3 & \frac{1-4k}{5}a_4 & \frac{1-5k}{6}a_5 & \\
x^2 & 0 & 1 & \frac{2-k}{3}a_1 & \frac{2-2k}{4}a_2 & \frac{2-3k}{5}a_3 & \frac{2-4k}{6}a_4 & \\
x^3 & 0 & 0 & 1 & \frac{3-k}{4}a_1 & \frac{3-2k}{5}a_2 & \frac{3-3k}{6}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{4-k}{5}a_1 & \frac{4-2k}{6}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{5-k}{6}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

(3)

64.4 Determinant representation of $e^{\sum_{i=1}^{n} a_i x^i}$

$$e^{\sum_{i=1}^{4} a_i x^i} =$$

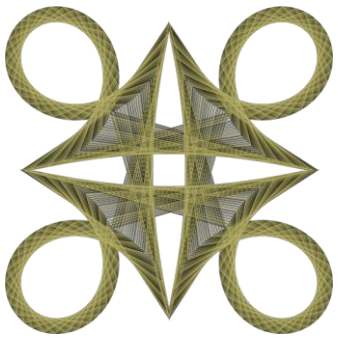
$$\begin{vmatrix}
1 & -a_1 & -a_2 & -a_3 & -a_4 & 0 & 0 & 0 & \rightarrow \\
x^1 & 1 & -\frac{1}{2}a_1 & -\frac{2}{3}a_2 & -\frac{3}{4}a_3 & -\frac{4}{5}a_4 & 0 & 0 & \\
x^2 & 0 & 1 & -\frac{1}{3}a_1 & -\frac{2}{4}a_2 & -\frac{3}{5}a_3 & -\frac{4}{6}a_4 & 0 & \\
x^3 & 0 & 0 & 1 & -\frac{1}{4}a_1 & -\frac{2}{5}a_2 & -\frac{3}{6}a_3 & -\frac{4}{7}a_4 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{1}{5}a_1 & -\frac{2}{6}a_2 & -\frac{3}{7}a_3 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{6}a_1 & -\frac{2}{7}a_2 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{7}a_1 & \\
x^7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & & \searrow
\end{vmatrix}$$

(4)

64.5 Determinant representation of $1+\ln\left(1+\sum_{i=1}^{n} a_i x^i\right)$

$$1+\ln\left(1+\sum_{i=1}^{\infty} a_i x^i\right)=$$

$$\begin{vmatrix}
1 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & \rightarrow \\
x^1 & 1 & \frac{1}{2}a_1 & \frac{1}{3}a_2 & \frac{1}{4}a_3 & \frac{1}{5}a_4 & \frac{1}{6}a_5 & \\
x^2 & 0 & 1 & \frac{2}{3}a_1 & \frac{2}{4}a_2 & \frac{2}{5}a_3 & \frac{2}{6}a_4 & \\
x^3 & 0 & 0 & 1 & \frac{3}{4}a_1 & \frac{3}{5}a_2 & \frac{3}{6}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{4}{5}a_1 & \frac{4}{6}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{5}{6}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} \qquad \textbf{(5)}$$



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64.6 Details to equ. (3)

The right side of equ. (3) is an infinite determinant of an finite polynomial in x. The result of the calculation of this determinant is for $k \in \mathbb{N}$ a finite, otherwise an infinite polynomial in x. The necessary stop condition is already in the Laplace expansion formula incorporated.

Example 1

$$\left(1+\sum_{i=1}^{n} a_i x^i\right)^2 =$$

$$\begin{vmatrix}
1 & -2a_1 & -2a_2 & -2a_3 & -2a_4 & -2a_5 & -2a_6 & \rightarrow \\
x^1 & 1 & -\frac{1}{2}a_1 & -\frac{3}{3}a_2 & -\frac{5}{4}a_3 & -\frac{7}{5}a_4 & -\frac{9}{6}a_5 & \\
x^2 & 0 & 1 & -\frac{0}{3}a_1 & -\frac{2}{4}a_2 & -\frac{4}{5}a_3 & -\frac{6}{6}a_4 & \\
x^3 & 0 & 0 & 1 & -\frac{-1}{4}a_1 & -\frac{1}{5}a_2 & -\frac{3}{6}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{-2}{5}a_1 & -\frac{0}{6}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{-3}{6}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

(6.1)

Special case n=1, k=2

$$(1+a_1x)^2 = 1+2a_1x+(a_1x)^2$$

The evaluation of the associated determinant along the first column (the x-column)

$$\begin{vmatrix}
1 & -2a_1 & 0 & 0 & 0 & 0 & 0 & \rightarrow \\
x^1 & 1 & -\frac{1}{2}a_1 & 0 & 0 & 0 & 0 & \\
x^2 & 0 & 1 & -\frac{0}{3}a_1 & 0 & 0 & 0 & \\
x^3 & 0 & 0 & 1 & -\frac{-1}{4}a_1 & 0 & 0 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{-2}{5}a_1 & 0 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{-3}{6}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

$$= 1+2a_1x+(a_1x)^2$$

provides resulting from the stop-0*s* the same result, a finite sequence.

(6.2)

Example 2

$$\left(1+\sum_{i=1}^{n} a_i x^i\right)^{\frac{1}{2}} =$$

$$\begin{vmatrix}
1 & -\frac{1}{2}a_1 & -\frac{1}{2}a_2 & -\frac{1}{2}a_3 & -\frac{1}{2}a_4 & -\frac{1}{2}a_5 & -\frac{1}{2}a_6 & \rightarrow \\
x^1 & 1 & -\frac{-1}{4}a_1 & -\frac{0}{6}a_2 & -\frac{1}{8}a_3 & -\frac{2}{10}a_4 & -\frac{3}{12}a_5 & \\
x^2 & 0 & 1 & -\frac{-3}{6}a_1 & -\frac{-2}{8}a_2 & -\frac{-1}{10}a_3 & -\frac{0}{12}a_4 & \\
x^3 & 0 & 0 & 1 & -\frac{-5}{8}a_1 & -\frac{-4}{10}a_2 & -\frac{-3}{12}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{-7}{10}a_1 & -\frac{-6}{12}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{-9}{12}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

(7.1)

Special case n=1, k=1/2

$$(1+a_1)^{\frac{1}{2}} =$$

$$\begin{vmatrix}
1 & -\frac{1}{2}a_1 & 0 & 0 & 0 & 0 & 0 & \rightarrow \\
x^1 & 1 & -\frac{-1}{4}a_1 & 0 & 0 & 0 & 0 & \\
x^2 & 0 & 1 & -\frac{-3}{6}a_1 & 0 & 0 & 0 & \\
x^3 & 0 & 0 & 1 & -\frac{-5}{8}a_1 & 0 & 0 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{-7}{10}a_1 & 0 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{-9}{12}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} =$$

$$1 + \frac{1}{2}a_1x - \frac{1}{8}(a_1x)^2 + \frac{1}{16}(a_1x)^3 - \frac{5}{128}(a_1x)^4 + \frac{7}{256}(a_1x)^5 \mp \ldots$$

Here we have a completely other situation. The blue 0s have no stop effect on the infinite determinant calculation, the result is an infinite series.

(7.2)

Example 3

Another nonstop without 0s.

$$\left(1+\sum_{i=1}^{n} a_i x^i\right)^{-\frac{1}{2}} =$$

$$\begin{vmatrix}
1 & \frac{1}{2}a_1 & \frac{1}{2}a_2 & \frac{1}{2}a_3 & \frac{1}{2}a_4 & \frac{1}{2}a_5 & \frac{1}{2}a_6 & \rightarrow \\
x^1 & 1 & \frac{3}{4}a_1 & \frac{4}{6}a_2 & \frac{5}{8}a_3 & \frac{6}{10}a_4 & \frac{7}{12}a_5 & \\
x^2 & 0 & 1 & \frac{5}{6}a_1 & \frac{6}{8}a_2 & \frac{7}{10}a_3 & \frac{8}{12}a_4 & \\
x^3 & 0 & 0 & 1 & \frac{7}{8}a_1 & \frac{8}{10}a_2 & \frac{9}{12}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{9}{10}a_1 & \frac{10}{12}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{11}{12}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

(8)

64.6 More details to equ. (4) can you find in Unit 6

64.7 Our targets are now determinant representations of

**** sinx, cosx, tanx**

**** Gudermannian function**

$gd(x) = \tan^{-1}(\sinh(x))$

$gd^{-1}(x) = \ln(\tan x + 1/\cos x)$

**** Bessel functions, f.i.**

$$I_0(x) = \sum_{i=0}^{\infty} \frac{1}{(i!)^2} \left(\frac{x}{2} \right)^{2i}$$

**** Riemann's Zeta function**

$$\zeta(x) = \sum_{i=1}^{\infty} \frac{1}{i^x}$$

with $x \rightarrow 1 + \sum_{i=1}^{n} a_i x^i$

This is your turn, my students!