Unit 63
Differential calculus
The Pro&Quo rule in determinant representation

Ch 12
A formula hunter‘s dream
Differential calculus, the Pro&Quo rule in determinant representation

And then appears on Puck’s iPad for the blink of an eye this wonderful formula, a generalized product&quotient rule in chevron determinant representation

$$\frac{d\left(\prod_{i=1}^{n}\frac{f_i(x)}{g_i(x)}\right)}{dx}=$$

$$\frac{1}{\prod_{i=1}^{n}G_i^2}*\begin{vmatrix} F_1' & G_1' & F_2' & G_2' & F_3' & G_3' & \dots & F_n' & G_n' \\ F_1 & G_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & G_1 & F_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & F_2 & G_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & G_2 & F_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & F_3 & G_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & F_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & F_n & G_n \end{vmatrix}$$

$$F_i=f_i(x)\ ;\ G_i=g_i(x)\ ;\ F_i'=\frac{df_i(x)}{dx}\ ;\ G_i'=\frac{dg_i(x)}{dx}$$

(1.1)

But, can one trust such an inspiration?
Confidence is good, control is better.
At first let’s check a special case,
special case 1

$$\begin{cases} f_i(x) = \begin{cases} f_2(x) & \text{for } i = 2 \\ 1 & i \neq 2 \end{cases} \\ g_i(x) = \begin{cases} g_2(x) & \text{for } i = 2 \\ 1 & i \neq 2 \end{cases} \end{cases} . \qquad (2.1)$$

We obtain here

$$\frac{d\left(\frac{f_2(x)}{g_2(x)}\right)}{dx} =$$

$$\frac{1}{G_2^2} * \begin{vmatrix} 0 & 0 & F_2' & G_2' & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & F_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & F_2 & G_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & G_2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{vmatrix} =$$

$$\frac{1}{G_2^2} * \begin{vmatrix} F_2' & G_2' \\ F_2 & G_2 \end{vmatrix} = \frac{-1}{G_2^2} * \begin{vmatrix} F_2 & G_2 \\ F_2' & G_2' \end{vmatrix}$$

and after the substitution

$F_2 = f(x)$; $G_2 = g(x)$; $F_2' = \frac{df(x)}{dx}$; $G_2' = \frac{dg(x)}{dx}$ finally

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{1}{(g(x))^2} * \begin{vmatrix} \frac{df(x)}{dx} & \frac{dg(x)}{dx} \\ f(x) & g(x) \end{vmatrix}$$

$$= \frac{-1}{(g(x))^2} * \begin{vmatrix} f(x) & g(x) \\ \frac{df(x)}{dx} & \frac{dg(x)}{dx} \end{vmatrix} . \qquad (2.2)$$

This is the well known quotient rule of the differential calculus in determinant representation,

This conclusion is correct. Go now to the next, special case 2

$$\begin{cases} f_i(x) = \begin{cases} f_1(x) & i = 1 \\ f_2(x) \text{ for } & i = 2 \\ 1 & i \neq 1,2 \end{cases} \\ (\forall i) g_i(x) = 1 \end{cases} \qquad (3.1)$$

resulting in

$$\frac{d\left(f_1(x)*f_2(x)\right)}{dx} =$$

$$\begin{vmatrix} F_1' & 0 & F_2' & 0 & 0 & 0 & \dots & 0 & 0 \\ F_1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & F_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & F_2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} F_1' & 0 & F_2' \\ F_1 & 1 & 0 \\ 0 & 1 & F_2 \end{vmatrix} = F_1F_2' + F_1'F_2$$

and with the substitution

$$F_1 = f_1(x)\ ;\ F_2 = f_2(x)\ ;\ F_1' = \frac{df_1(x)}{dx}\ ;\ F_2' = \frac{df_2(x)}{dx}$$

finally in

$$\frac{d\left(f_1(x)*f_2(x)\right)}{dx} = f_1(x)\frac{df_2(x)}{dx} + \frac{df_1(x)}{dx}f_2(x)\ ,$$

(3.2)

the associated product rule, there is also no contradiction.

More, special case 3

$$\begin{cases} f_i(x) = \begin{cases} f_1(x) & i = 1 \\ f_2(x) \text{ for } & i = 2 \\ 1 & i \neq 1,2 \end{cases} \\ g_i(x) = \begin{cases} g_1(x) & i = 1 \\ 1 \text{ for} & i \neq 1 \end{cases} \end{cases} \tag{4.1}$$

Result

$$\frac{d\left(\frac{f_1(x)*f_2(x)}{g_1(x)}\right)}{dx} = \frac{1}{\left(g_1(x)\right)^2}\begin{vmatrix} \frac{df_1(x)}{dx} & \frac{dg_1(x)}{dx} & \frac{df_2(x)}{dx} \\ f_1(x) & g_1(x) & 0 \\ 0 & g_1(x) & f_2(x) \end{vmatrix}$$

$$= \frac{1}{\left(g_1(x)\right)^2}\left(\begin{vmatrix} \frac{df_1(x)}{dx} & \frac{dg_1(x)}{dx} \\ f_1(x) & g_1(x) \end{vmatrix} f_2(x) + f_1(x)g_1(x)\frac{df_2(x)}{dx}\right) \tag{4.2}$$

Also correct.

The expression of the determinant in equ. (1.1) delivers

$$\frac{d\left(\prod_{i=1}^{n}\frac{f_i(x)}{g_i(x)}\right)}{dx}=$$

$$\frac{1}{\prod_{i=1}^{n}G_i^2}*\begin{vmatrix} F_1' & G_1' & F_2' & G_2' & F_3' & G_3' & \dots & F_n' & G_n' \\ F_1 & G_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & G_1 & F_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & F_2 & G_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & G_2 & F_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & F_3 & G_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & F_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & F_n & G_n \end{vmatrix} \qquad (1.1)$$

$$= \frac{1}{\prod_{i=1}^{n} G_i^2} * \left(\begin{vmatrix} F_1' & G_1' \\ F_1 & G_1 \end{vmatrix} (F_2 G_2)(F_3 G_3)...(F_n G_n) \right.$$

$$+ (F_1 G_1) \begin{vmatrix} F_2' & G_2' \\ F_2 & G_2 \end{vmatrix} (F_3 G_3)...(F_n G_n)$$

$$+ (F_1 G_1)(F_2 G_2 \begin{vmatrix} F_3' & G_3' \\ F_3 & G_3 \end{vmatrix} (F_4 G_4)...(F_n G_n)$$

$$\left. + ... + (F_1 G_1)(F_2 G_2)(F_3 G_3)...(F_{n-1} G_{n-1}) \begin{vmatrix} F_2' & G_2' \\ F_2 & G_2 \end{vmatrix} \right)$$

$$F_i = f_i(x) \; ; \; G_i = g_i(x) \; ; \; F_i' = \frac{df_i(x)}{dx} \; ; \; G_i' = \frac{dg_i(x)}{dx}$$

(1.2)

Now we start the **proof of the Pro&Quo formula (1)** by mathematical induction [2].
This is your turn, my student.
And also to think about applications of equ. (1), esp. in physics and engineering.

[1] http://www.foodhuntermark.com/
[2] http://en.wikipedia.org/wiki/Mathematical_induction

Ch 14
Even more hunting
14.1
From Ch 12, equ. (1) follows

$$\frac{d\left(\frac{f_1(x)}{g_1(x)} * \frac{f_2(x)}{g_2(x)}\right)}{dx} = \frac{1}{(G_1 G_2)^2} * \begin{vmatrix} F_1' & G_1' & F_2' & G_2' \\ F_1 & G_1 & 0 & 0 \\ 0 & G_1 & F_2 & 0 \\ 0 & 0 & F_2 & G_2 \end{vmatrix}$$

$$= \frac{1}{(G_1 G_2)^2} * \left((F_1 G_1) \begin{vmatrix} F_2' & G_2' \\ F_2 & G_2 \end{vmatrix} + \begin{vmatrix} F_1' & G_1' \\ F_1 & G_1 \end{vmatrix} (F_2 G_2) \right)$$

$$F_i = f_i(x) \; ; \; G_i = g_i(x) \; ; \; F_i' = \frac{df_i(x)}{dx} \; ; \; G_i' = \frac{dg_i(x)}{dx}$$

(5.1)

Find a corresponding one-determinant representation of the second derivative

$$\frac{d^2\left(\frac{f_1(x)}{g_1(x)} * \frac{f_2(x)}{g_2(x)}\right)}{2!dx^2} =$$

$$\frac{1}{(G_1G_2)^3}(F_1F_2^{//}G_1G_1G_2G_2 + F_1F_2G_1^{/}G_1^{/}G_2G_2$$
$$+ F_1F_2G_1G_1^{/}G_2G_2^{/} + F_1F_2G_1G_1G_2^{/}G_2^{/}$$
$$+ F_1^{/}F_2^{/}G_1G_1G_2G_2 + F_1^{//}F_2G_1G_1G_2G_2$$
$$- F_1F_2^{/}G_1G_1^{/}G_2G_2 - F_1F_2G_1G_1^{//}G_2G_2$$
$$- F_1F_2^{/}G_1G_1G_2G_2^{/} - F_1F_2G_1G_1G_2G_2^{//}$$
$$- F_1^{/}F_2G_1G_1^{/}G_2G_2 - F_1^{/}F_2G_1G_1G_2G_2^{/})$$

$$= \frac{1}{(G_1G_2)^3}(F_1\begin{vmatrix} F_2 & G_2 & 0 \\ F_2^{/} & G_2^{/} & G_2 \\ F_2^{//} & G_2^{//} & G_2^{/} \end{vmatrix} G_1G_1$$
$$+ \begin{vmatrix} F_1 & G_1 \\ F_1^{/} & G_1^{/} \end{vmatrix} * \begin{vmatrix} F_2 & G_2 \\ F_2^{/} & G_2^{/} \end{vmatrix} G_1G_2$$
$$+ \begin{vmatrix} F_1 & G_1 & 0 \\ F_1^{/} & G_1^{/} & G_1 \\ F_1^{//} & G_1^{//} & G_1^{/} \end{vmatrix} F_2G_2G_2)$$

$$F = f(x)\ ,\ F^{/} = \frac{df(x)}{dx}\frac{}{1!}\ ,\ F^{//} = \frac{d^2f(x)}{dx^2}\frac{}{2!}$$

Check it! (5.2)

and the one-determinant representations for the extensions

$$\frac{d^2\left(\prod_{i=1}^{n}\frac{f_i(x)}{g_i(x)}\right)}{2!dx^2}\ ,\ \frac{d^n\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)}{n!dx^n}\ ,$$

$$\text{and}\ \frac{d^n\left(\frac{f_1(x)}{g_1(x)}*\frac{f_2(x)}{g_2(x)}\right)}{n!dx^n}\ .\qquad (5.3,4.1,5)$$

Hint to equ. (5.4.1):

1** Sum-determinant representation for
n=0,1,2,3,4

$$\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)=\frac{F_1}{G}F_2$$

$$\frac{d\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)}{1!dx}=\frac{F_1}{G}F_2'-\frac{\begin{vmatrix}F_1 & G\\ F_1' & G'\end{vmatrix}}{G^2}F_2$$

$$\frac{d^2\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)}{2!dx^2}=\frac{F_1}{G}F_2''-\frac{\begin{vmatrix}F_1 & G\\ F_1' & G'\end{vmatrix}}{G^2}F_2'$$

$$+\frac{\begin{vmatrix}F_1 & G & 0\\ F_1' & G' & G\\ F_1'' & G'' & G'\end{vmatrix}}{G^3}F_2$$

$$\frac{d^3\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)}{3!dx^3}=\frac{F_1}{G}F_2'''-\frac{\begin{vmatrix}F_1 & G\\ F_1' & G'\end{vmatrix}}{G^2}F_2''$$

$$+\frac{\begin{vmatrix}F_1 & G & 0\\ F_1' & G' & G\\ F_1'' & G'' & G'\end{vmatrix}}{G^3}F_2'-\frac{\begin{vmatrix}F_1 & G & 0 & 0\\ F_1' & G' & G & 0\\ F_1'' & G'' & G' & G\\ F_1''' & G''' & G'' & G'\end{vmatrix}}{G^4}F_2$$

$$\frac{d^4\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)}{4!dx^3} = \frac{F_1}{G}F_2^{////} - \frac{\begin{vmatrix} F_1 & G \\ F_1^{/} & G^{/} \end{vmatrix}}{G^2}F_2^{///}$$

$$+\frac{\begin{vmatrix} F_1 & G & 0 \\ F_1^{/} & G^{/} & G \\ F_1^{//} & G^{//} & G^{/} \end{vmatrix}}{G^3}F_2^{//} - \frac{\begin{vmatrix} F_1 & G & 0 & 0 \\ F_1^{/} & G^{/} & G & 0 \\ F_1^{//} & G^{//} & G^{/} & G \\ F_1^{///} & G^{///} & G^{//} & G^{/} \end{vmatrix}}{G^4}F_2^{/}$$

$$+\frac{\begin{vmatrix} F_1 & G & 0 & 0 & 0 \\ F_1^{/} & G^{/} & G & 0 & 0 \\ F_1^{//} & G^{//} & G^{/} & G & 0 \\ F_1^{///} & G^{///} & G^{//} & G^{/} & G \\ F_1^{////} & G^{////} & G^{///} & G^{//} & G^{/} \end{vmatrix}}{G^5}F_2$$

(5.4.2)

Für x = 0 ist damit zugleich auch eine Summen-Determinanten Notation für die

McLaurinreihe von $f(x) = \frac{f_1(x)*f_2(x)}{g(x)}$

gegeben:

$$f(x) = \frac{f_1(x)*f_2(x)}{g(x)} = \sum_{i=0}^{\infty} \left(\frac{d^i \left(\frac{f_1(x)*f_2(x)}{g(x)} \right)}{i!dx^i}; \; x=0 \right) * x^i \qquad (5.4.3)$$

Example 1

$f_1(x) = e^x$, $f_2(x) = \sin x$, $g(x) = \cos x$ (5.4.3.1)

$x = 0 \rightarrow$

$$F_1^{(i)} = \frac{1}{i!}$$

$$F_2^{(2i)} = 0, \quad F_2^{(1)} = \frac{+1}{1!},\ F_2^{(3)} = \frac{-1}{3!},\ F_2^{(5)} = \frac{+1}{5!},$$

$$F_2^{(7)} = \frac{-1}{7!},\ F_2^{(9)} = \frac{+1}{9!},\ F_2^{(11)} = \frac{-1}{11!},\ \ldots$$

$$G^{(2i+1)} = 0,\ G = \frac{+1}{0!},\ G^{(2)} = \frac{-1}{2!},\ G^{(4)} = \frac{+1}{4!},$$

$$G^{(6)} = \frac{-1}{6!},\ G^{(8)} = \frac{+1}{8!},\ G^{(10)} = \frac{-1}{10!},\ \ldots$$

(5.4.3.2)

$x = 0 \rightarrow$

$$\left(\frac{f_1(x)*f_2(x)}{g_1(x)}\right)=\frac{\frac{1}{0!}}{\frac{1}{0!}}0=0$$

$$\frac{d\left(\frac{f_1(x)*f_2(x)}{g_1(x)}\right)}{1!dx}=\frac{\frac{1}{0!}}{\frac{1}{0!}}\frac{1}{1!}-\frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!}\\ \frac{1}{1!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^2}0=1$$

$$\frac{d^2\left(\frac{f_1(x)*f_2(x)}{g_1(x)}\right)}{2!dx^2}=\frac{\frac{1}{0!}}{\frac{1}{0!}}0-\frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!}\\ \frac{1}{1!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^2}\frac{1}{1!}$$

$$+\frac{\begin{vmatrix} \frac{1}{0!} & \frac{1}{0!} & 0 \\ \frac{1}{1!} & 0 & \frac{1}{0!} \\ \frac{1}{2!} & \frac{-1}{2!} & 0 \end{vmatrix}}{\left(\frac{1}{0!}\right)^3} 0 = \frac{\left|\frac{1}{1!}\right|}{\left(\frac{1}{0!}\right)^2}\left(\frac{1}{1!}\right) = 1$$

$$\frac{d^3\left(\frac{f_1(x)*f_2(x)}{g_1(x)}\right)}{3!dx^3} = \frac{\frac{1}{0!}}{\frac{1}{0!}}\frac{-1}{3!} - \frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!}\\ \frac{1}{1!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^2}0$$

$$+\frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!} & 0\\ \frac{1}{1!} & 0 & \frac{1}{0!}\\ \frac{1}{2!} & \frac{-1}{2!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^3}\frac{1}{1!} - \frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!} & 0 & 0\\ \frac{1}{1!} & 0 & \frac{1}{0!} & 0\\ \frac{1}{2!} & \frac{-1}{2!} & 0 & \frac{1}{0!}\\ \frac{1}{3!} & 0 & \frac{-1}{2!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^4}0 =$$

$$\frac{\left|\frac{1}{0!}\right|}{\left(\frac{1}{0!}\right)^1}\left(\frac{-1}{3!}\right) - \frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!}\\ \frac{1}{2!} & \frac{-1}{2!}\end{vmatrix}}{\left(\frac{1}{0!}\right)^3}\left(\frac{1}{1!}\right) = \frac{5}{6}$$

$$\frac{d^4\left(\frac{f_1(x)*f_2(x)}{g_1(x)}\right)}{4!dx^4} = \frac{\frac{1}{0!}}{\frac{1}{0!}}0 - \frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!}\\ \frac{1}{1!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^2}\frac{-1}{3!}$$

$$+\frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!} & 0\\ \frac{1}{1!} & 0 & \frac{1}{0!}\\ \frac{1}{2!} & \frac{-1}{2!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^3}0 - \frac{\begin{vmatrix}\frac{1}{0!} & \frac{1}{0!} & 0 & 0\\ \frac{1}{1!} & 0 & \frac{1}{0!} & 0\\ \frac{1}{2!} & \frac{-1}{2!} & 0 & \frac{1}{0!}\\ \frac{1}{3!} & 0 & \frac{-1}{2!} & 0\end{vmatrix}}{\left(\frac{1}{0!}\right)^4}\frac{1}{1!}$$

$$+\frac{\begin{vmatrix} \frac{1}{0!} & \frac{1}{0!} & 0 & 0 & 0 \\ \frac{1}{1!} & 0 & \frac{1}{0!} & 0 & 0 \\ \frac{1}{2!} & \frac{-1}{2!} & 0 & \frac{1}{0!} & 0 \\ \frac{1}{3!} & 0 & \frac{-1}{2!} & 0 & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{4!} & 0 & \frac{-1}{2!} & 0 \end{vmatrix}}{\left(\frac{1}{0!}\right)^5} 0 =$$

$$\frac{\begin{vmatrix} \frac{1}{1!} \end{vmatrix}}{\left(\frac{1}{0!}\right)^2}\left(\frac{-1}{3!}\right)-\frac{\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{-1}{2!} \end{vmatrix}}{\left(\frac{1}{0!}\right)^4}\left(\frac{1}{1!}\right)=\frac{1}{2}$$

(5.4.3.3)

More!

$\rightarrow$ McLaurin series representation of

$$\frac{e^x * \sin x)}{\cos x} = x + x^2 + \frac{5}{6}x^3 + \frac{1}{2}x^4 + \left(\frac{41}{120}x^5 + \frac{71}{360}x^6 + \ldots\right) \qquad (5.4.3.4)$$

Example 2

McLaurin series in sum-determinant and determinant-product representation of

$$\frac{f_1(x)*f_2(x)}{g(x)} \text{ with}$$

$$(\forall i)F_2^{(2i)} = 0$$

$$G = 1\ ;\ (\forall i)G^{(2i+1)} = 0 \qquad (5.5.1)$$

$$\frac{f_1(x)*f_2(x)}{g(x)} = \left|F_1\right|F_2'x + \left|F_1'\right|F_2'x^2 + \left(\left|F_1\right|F_2''' - \begin{vmatrix} F_1 & 1 \\ F_1'' & G'' \end{vmatrix}F_2'\right)x^3 + \left(\left|F_1'\right|F_2''' - \begin{vmatrix} F_1' & 1 \\ F_1''' & G'' \end{vmatrix}F_2'\right)x^4 +$$

$$\left(\left|F_1\right|F_2^{(5)}-\begin{vmatrix}F_1 & 1\\ F_1^{//} & G^{//}\end{vmatrix}F_2^{///}+\begin{vmatrix}F_1 & 1 & 0\\ F_1^{//} & G^{//} & 1\\ F_1^{////} & G^{////} & G^{//}\end{vmatrix}F_2^{/}\right)x^5+$$

$$\left(\left|F_1^{/}\right|F_2^{(5)}-\begin{vmatrix}F_1^{/} & 1\\ F_1^{///} & G^{//}\end{vmatrix}F_2^{///}+\begin{vmatrix}F_1^{/} & 1 & 0\\ F_1^{///} & G^{//} & 1\\ F_1^{(5)} & G^{////} & G^{//}\end{vmatrix}F_2^{/}\right)x^6+$$

$$\left(\left|F_1\right|F_2^{(7)}-\begin{vmatrix}F_1 & 1\\ F_1^{//} & G^{//}\end{vmatrix}F_2^{(5)}+\begin{vmatrix}F_1 & 1 & 0\\ F_1^{//} & G^{//} & 1\\ F_1^{////} & G^{////} & G^{//}\end{vmatrix}F_2^{///}+\begin{vmatrix}F_1 & 1 & 0 & 0\\ F_1^{//} & G^{//} & 1 & 0\\ F_1^{////} & G^{////} & G^{//} & 1\\ F_1^{(6)} & G^{(6)} & G^{////} & G^{//}\end{vmatrix}F_2^{/}\right)x^7+$$

$$\left(\left|F_1^{/}\right|F_2^{(7)}-\begin{vmatrix}F_1^{/} & 1\\ F_1^{///} & G^{//}\end{vmatrix}F_2^{(5)}+\begin{vmatrix}F_1^{/} & 1 & 0\\ F_1^{///} & G^{//} & 1\\ F_1^{(5)} & G^{////} & G^{//}\end{vmatrix}F_2^{///}+\begin{vmatrix}F_1^{/} & 1 & 0 & 0\\ F_1^{///} & G^{//} & 1 & 0\\ F_1^{(5)} & G^{////} & G^{//} & 1\\ F_1^{(7)} & G^{(6)} & G^{////} & G^{//}\end{vmatrix}F_2^{/}\right)x^8+\ldots$$

(5.5.2)

$$\frac{f_1(x)*f_2(x)}{g(x)}=\left(\frac{f_1(x)}{g(x)}\right)*f_2(x)=$$

$$\begin{vmatrix} 0 & 1 & x^2 & x^4 & x^6 & \rightarrow \\ F_1+F_1'x & 1 & 0 & 0 & 0 & \\ F_1''+F_1'''x & G'' & 1 & 0 & 0 & \\ F_1''''+F_1^{(5)}x & G'''' & G'' & 1 & 0 & \\ F_1^{(6)}+F_1^{(7)}x & G^{(6)} & G'''' & G'' & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix} *$$

$$\begin{vmatrix} 0 & 1 & x^2 & x^4 & x^6 & \rightarrow \\ F_2'x & 1 & 0 & 0 & 0 & \\ F_2'''x & 0 & 1 & 0 & 0 & \\ F_2^{(5)}x & 0 & 0 & 1 & 0 & \\ F_2^{(7)}x & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix} \qquad (5.5.3)$$

2** **The one-determinant representation of the equs (5.4.2) and (5.5) is your turn, my student.**

Hint:

$$\frac{d^4\left(\frac{f_1(x)*f_2(x)}{g(x)}\right)}{4!dx^3} =$$

$$-\frac{\begin{vmatrix} 0 & F_2^{////} & F_2^{///} & F_2^{//} & F_2^{/} & F_2 \\ F_1 & G & 0 & 0 & 0 & 0 \\ F_1^{/} & G^{/} & G & 0 & 0 & 0 \\ F_1^{//} & G^{//} & G^{/} & G & 0 & 0 \\ F_1^{///} & G^{///} & G^{//} & G^{/} & G & 0 \\ F_1^{////} & G^{////} & G^{///} & G^{//} & G^{/} & G \end{vmatrix}}{\begin{vmatrix} G & 0 & 0 & 0 & 0 \\ G^{/} & G & 0 & 0 & 0 \\ G^{//} & G^{/} & G & 0 & 0 \\ G^{///} & G^{//} & G^{/} & G & 0 \\ G^{////} & G^{///} & G^{//} & G^{/} & G \end{vmatrix} = G^5} \qquad (5.4.4)$$

14.2

A simple transformation

$$\left(\frac{f_1(x)}{g_1(x)} * \frac{f_2(x)}{g_2(x)}\right)' = \left(\frac{f_1(x)}{g_1(x) * \frac{g_2(x)}{f_2(x)}}\right)' =$$

$$\frac{1}{\left(g_1(x) * g_2(x)\right)^2} \begin{vmatrix} f_1'(x) & g_1'(x) & f_2'(x) & g_2'(x) \\ f_1(x) & g_1(x) & 0 & 0 \\ 0 & g_1(x) & f_2(x) & 0 \\ 0 & 0 & f_2(x) & g_2(x) \end{vmatrix}$$

(6.1)

$$\left(\frac{f_1(x)}{g_1(x)}*\frac{f_2(x)}{g_2(x)}*\frac{f_3(x)}{g_3(x)}\right)'=\left(\frac{f_1(x)}{g_1(x)*\dfrac{g_2(x)}{f_2(x)*\dfrac{f_3(x)}{g_3(x)}}}\right)'=$$

$$\frac{1}{\left(g_1(x)*g_2(x)*g_3(x)\right)^2}*$$

$$\begin{vmatrix} f_1'(x) & g_1'(x) & f_2'(x) & g_2'(x) & f_3'(x) & g_3'(x) \\ f_1(x) & g_1(x) & 0 & 0 & 0 & 0 \\ 0 & g_1(x) & f_2(x) & 0 & 0 & 0 \\ 0 & 0 & f_2(x) & g_2(x) & 0 & 0 \\ 0 & 0 & 0 & g_2(x) & f_3(x) & 0 \\ 0 & 0 & 0 & 0 & f_3(x) & g_3(x) \end{vmatrix}$$

(6.2)

indicates a structural relationship of this formulas with continued fractions and their determinant representation:

$$\frac{a}{b} = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \ddots + \cfrac{b_n}{a_n}}}}}$$

$$= \frac{\begin{vmatrix} a_0 & b_1 & 0 & 0 & 0 & 0 & 0 \\ -1 & a_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & a_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_3 & b_4 & \ddots & 0 \\ 0 & 0 & 0 & -1 & a_4 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & b_n \\ 0 & 0 & 0 & 0 & 0 & -1 & a_n \end{vmatrix}}{\begin{vmatrix} a_1 & b_2 & 0 & 0 & 0 & 0 \\ -1 & a_2 & b_3 & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_4 & \ddots & 0 \\ 0 & 0 & -1 & a_4 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & b_n \\ 0 & 0 & 0 & 0 & -1 & a_n \end{vmatrix}} \qquad (7)$$

and opens the door to a lot of laterals, f.i.
1. Extension from *

$$\left(\frac{f_1(x)}{g_1(x)} * \frac{f_2(x)}{g_2(x)} * \frac{f_3(x)}{g_3(x)}\right)' = \left(\frac{f_1(x)}{g_1(x) * \frac{g_2(x)}{f_2(x) * \frac{f_3(x)}{g_3(x)}}}\right)' =$$

$$\frac{1}{\left(g_1(x) * g_2(x) * g_3(x)\right)^2} *$$

$$\begin{vmatrix} f_1'(x) & g_1'(x) & f_2'(x) & g_2'(x) & f_3'(x) & g_3'(x) \\ f_1(x) & g_1(x) & 0 & 0 & 0 & 0 \\ 0 & g_1(x) & f_2(x) & 0 & 0 & 0 \\ 0 & 0 & f_2(x) & g_2(x) & 0 & 0 \\ 0 & 0 & 0 & g_2(x) & f_3(x) & 0 \\ 0 & 0 & 0 & 0 & f_3(x) & g_3(x) \end{vmatrix}$$

to +

$$\left(\cfrac{f_1(x)}{g_1(x) + \cfrac{g_2(x)}{f_2(x) + \cfrac{f_3(x)}{g_3(x)}}} \right)' = ??? \qquad (6.3)$$

2. Remarkable +/* continued fractions

2.1 in natural numbers

Start

$$\cfrac{2}{1\lozenge \cfrac{3}{2\lozenge \cfrac{4}{3\lozenge \cfrac{5}{4\lozenge \cfrac{6}{5\lozenge \cfrac{7}{6\lozenge \cfrac{8}{7\lozenge \cfrac{9}{\ddots}}}}}}}} \rightarrow \begin{cases} 1 \\ ? \dfrac{\pi}{2} ? \end{cases} \text{ for } \lozenge = \begin{cases} + \\ * \end{cases} \qquad (8.1)$$

Change the lobsters

$$\frac{1}{2\Diamond\dfrac{2}{3\Diamond\dfrac{3}{4\Diamond\dfrac{4}{5\Diamond\dfrac{5}{6\Diamond\dfrac{6}{7\Diamond\dfrac{7}{8\Diamond\frac{8}{\ldots}}}}}}}} \rightarrow \begin{cases} \dfrac{3-e}{e-2} & \text{for } \Diamond = \begin{cases} + \\ * \end{cases} \\ ? \end{cases} \tag{8.2}$$

And in between

$$\frac{1}{1\Diamond\dfrac{2}{2\Diamond\dfrac{3}{3\Diamond\dfrac{4}{4\Diamond\dfrac{5}{5\Diamond\dfrac{6}{6\Diamond\dfrac{7}{7\Diamond\frac{8}{\ldots}}}}}}}} \rightarrow \begin{cases} \dfrac{1}{e-1} & \text{for } \Diamond = \begin{cases} + \\ * \end{cases} \\ ? \end{cases} \tag{8.3}$$

2.2 in odd numbers

Start

$$\cfrac{3}{1\Diamond\cfrac{5}{3\Diamond\cfrac{7}{5\Diamond\cfrac{9}{7\Diamond\cfrac{11}{9\Diamond\cfrac{13}{11\Diamond\cfrac{15}{13\Diamond\cfrac{17}{\ddots}}}}}}}}} \rightarrow \begin{cases} ? \\ ? \end{cases} \text{ for } \Diamond = \begin{cases} + \\ * \end{cases} \tag{9.1}$$

Change the lobsters

$$\frac{1}{3\lozenge\,\frac{3}{5\lozenge\,\frac{5}{7\lozenge\,\frac{7}{9\lozenge\,\frac{9}{11\lozenge\,\frac{11}{13\lozenge\,\frac{13}{15\lozenge\,\frac{15}{\ldots}}}}}}}}} \rightarrow \begin{cases} ? \\ ? \end{cases} \text{ for } \lozenge = \begin{cases} + \\ * \end{cases}$$

(9.2)

And in between

$$\frac{1}{1\lozenge\,\frac{3}{3\lozenge\,\frac{5}{5\lozenge\,\frac{7}{7\lozenge\,\frac{9}{9\lozenge\,\frac{11}{11\lozenge\,\frac{13}{13\lozenge\,\frac{15}{\ldots}}}}}}}}} \rightarrow \begin{cases} ? \\ ? \end{cases} \text{ for } \lozenge = \begin{cases} + \\ * \end{cases}$$

(9.3)

2.3 Mixed odd-even

Start

$$\cfrac{3}{2\lozenge\cfrac{5}{4\lozenge\cfrac{7}{6\lozenge\cfrac{9}{8\lozenge\cfrac{11}{10\lozenge\cfrac{13}{12\lozenge\cfrac{15}{14\lozenge\cfrac{17}{\cdots}}}}}}}} \rightarrow \begin{cases} ? \\ ? \end{cases} \text{for } \lozenge = \begin{cases} + \\ * \end{cases} \tag{10.1}$$

Change the lobsters

$$\frac{2}{3\Diamond\dfrac{4}{5\Diamond\dfrac{6}{7\Diamond\dfrac{8}{9\Diamond\dfrac{10}{11\Diamond\dfrac{12}{13\Diamond\dfrac{14}{15\Diamond\dfrac{16}{\ldots}}}}}}}} \to \begin{cases} \dfrac{2-\sqrt{e}}{\sqrt{e}-1} \\ ? \end{cases} \text{ for } \Diamond = \begin{cases} + \\ * \end{cases}$$

(10.2)

2.4 in primes

Start

$$\frac{5}{3\Diamond\dfrac{7}{5\Diamond\dfrac{11}{11\Diamond\dfrac{13}{13\Diamond\dfrac{17}{17\Diamond\dfrac{19}{19\Diamond\dfrac{23}{23\Diamond\dfrac{29}{\ldots}}}}}}}} \to \begin{cases} ? \\ ? \end{cases} \text{ for } \Diamond = \begin{cases} + \\ * \end{cases}$$

(11.1)

Change the lobsters

$$\cfrac{3}{5\lozenge\cfrac{5}{7\lozenge\cfrac{7}{11\lozenge\cfrac{11}{13\lozenge\cfrac{13}{17\lozenge\cfrac{17}{19\lozenge\cfrac{19}{23\lozenge\frac{23}{\ldots}}}}}}}}} \rightarrow \begin{cases} ? \\ ? \end{cases} \text{ for } \lozenge = \begin{cases} + \\ * \end{cases}$$

(11.2)

And in between

$$\cfrac{3}{3\lozenge\cfrac{5}{5\lozenge\cfrac{7}{7\lozenge\cfrac{11}{11\lozenge\cfrac{13}{13\lozenge\cfrac{17}{17\lozenge\cfrac{19}{19\lozenge\frac{23}{\ldots}}}}}}}}} \rightarrow \begin{cases} ? \\ ? \end{cases} \text{ for } \lozenge = \begin{cases} + \\ * \end{cases}$$

(11.3)