# Unit 42
# A Math Fleet on the “Pascal´s Triangle and Determinant Patterns of Reciprocals of Factorials” Sea

Heavier-than-air flying machines are impossible.
Lord Kelvin, president of Royal Society (1895)

Pascal´s triangle (PT) and determinant patterns of reciprocals of factorials (DPRF) are 2D mathematical objects that cast a long shadow, see [1- 8].
Here is shown the connection of PT with structured sums of powers of integers and with Pi squared.
Continue to exhaust the megapotential of PT:

** PT and sums of powers of integers with ± oscillating Patterns, f.e.
$1^m+2^m-3^m-4^m+5^m+6^m-7^m-8^m+9^m+10^m-11^m-12^m\pm\ldots$
** PT and determinant patterns of reciprocals of factorials
** Find relations of PT with e, other mathematical constants, $\sqrt[m]{n}$ , $\Phi^n$, etc.
** PT and integer sequences, e.g. Bernoulli and Euler numbers
** PT and Taylor series of elementary functions
** PT and continued fractions
** PT and the sequence of primes
** PT and the ternary tree of Pythagorean triples

References


1 D.E. Knuth. The Art of Computer Programming. Vol. 1
Addison-Wesley 1997, ISBN 0-201-89683-4
p. 52-74

2 http://mathworld.wolfram.com/PascalsTriangle.html

3 http://mathworld.wolfram.com/BinomialCoefficient.html

4 S.Wolfram. A New Kind of Science.
Wolfram Media 2002, ISBN 1-57955-008-8

5 http://en.wikipedia.org/wiki/Pascal%27s_triangle

6 http://de.wikipedia.org/wiki/Pascalsches_Dreieck

7 http://en.wikipedia.org/wiki/Binomial_coefficient

8 http://de.wikipedia.org/wiki/Binomialkoeffizient

9 http://mathworld.wolfram.com/BernoulliNumber.html

10 http://mathworld.wolfram.com/EulerNumber.html

# Math Ship One

Pascal's triangle and the sums of powers of integers

$$\sum_{i=1}^{n} i^{m=6} = \frac{\begin{vmatrix} (-n)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$\sum_{i=1}^{n}(-1)^{n-i}\,i^{m=6} = \frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ n^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ n^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ n^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ n^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ n^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

(6+1) by (6+1) determinants

# Math Ship Two

Pascal's triangle, the sums of powers of integers, and Pi squared

$$\sum_{i=1}^{n} i^{m=6} = \frac{\begin{vmatrix} (-n)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$D(k=7,n)=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 3 & 1 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 6 & 1 & 0 & 0 \\ p^4 & 0 & 0 & 5 & 10 & 1 & 0 \\ p^5 & 0 & 0 & 1 & 15 & 15 & 1 \\ p^6 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i}i^{2}}=\frac{D(k,n)}{D(k,1)}$$

$$\sum_{i=1}^{n}(-1)^{n-i}i^{2}=\sum_{i=1}^{n}i^{1}=\frac{1}{2}p\ ;\ p=n(n+1)$$

$$\lim_{k\to\infty}\binom{2k+2}{2}\frac{D(k,0)}{D(k+1,0)}=\frac{\pi^2}{2}$$

Recurrence relation

$$D(k,n) = (-p)^{k-1} + \sum_{i=1}^{k-1} (-1)^{i-1} \binom{k}{k-2i} D(k-i,n)$$

$$\sum_{i=0}^{k} (-1)^{i} \binom{k}{k-2i} D(k-i,n) = (-p)^{k-1}$$

$$D(0,n) = 0, \quad D(1,n) = 1$$

Relation between PT $D(k,0)$ and determinant patterns of reciprocals of factorials

2073

1 1 3 17 155 ↑ 38227

↑ ↑ ↑ ↑ ↑ ↑ ↑

$$D(k=7,0) = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & 1 & 0 \\ 0 & 0 & 0 & 1 & 15 & 15 & 1 \\ 0 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant

$$\Leftrightarrow$$

$$\begin{array}{ccccccc} \frac{1}{2!} & \frac{1}{4!} & \frac{3}{6!} & \frac{17}{8!} & \frac{155}{10!} & \frac{2073}{12!} & \frac{38227}{14!} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

$$eb(k=7)=\begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \frac{1}{12!} & \frac{1}{14!} \\ \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \frac{1}{13!} \\ 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} \\ 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} \\ 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} \\ 0 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} \end{vmatrix}$$

$$\Leftrightarrow$$

$$2^{3}\frac{1}{4!}\qquad 2^{7}\frac{17}{8!}\qquad 2^{11}\frac{2073}{12!}$$

$$2^{1}\frac{1}{2!}\uparrow 2^{5}\frac{3}{6!}\uparrow 2^{9}\frac{155}{10!}\uparrow 2^{13}\frac{38227}{14!}$$

$$\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow$$

$$\mathrm{be}(k=7)=\begin{vmatrix}
\frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \frac{1}{13!} \\
\frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \frac{1}{12!} \\
0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} \\
0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} \\
0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} \\
0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!}
\end{vmatrix}$$

$$\Leftrightarrow$$

$$\frac{2^3-1}{2^4-1}\frac{1}{4!}\quad\frac{2^7-1}{2^8-1}\frac{17}{8!}\quad\frac{2^{11}-1}{2^{12}-1}\frac{2073}{12!}$$

$$\frac{2^1-1}{2^2-1}\frac{1}{2!}\uparrow\frac{2^5-1}{2^6-1}\frac{3}{6!}\uparrow\frac{2^9-1}{2^{10}-1}\frac{155}{10!}\uparrow\frac{2^{13}-1}{2^{14}-1}\frac{38227}{14!}$$

$$\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow$$

$$b(k=7)=\begin{vmatrix}
\frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \frac{1}{13!} & \frac{1}{15!} \\
\frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \frac{1}{13!} \\
0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} \\
0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} \\
0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} \\
0 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!}
\end{vmatrix}$$

$$\Leftrightarrow$$

$$-\delta\left(\mathrm{m}=\begin{matrix}2\mathrm{k}-1=9\\ 2\mathrm{k}\end{matrix}\right)=$$

$$\begin{matrix}1 & -2 & 16 & -272 & 7936\\ \frac{(-4)^0}{1}1 & \frac{(-4)^1}{2}1 & \frac{(-4)^2}{3}3 & \frac{(-4)^3}{4}17 & \frac{(-4)^4}{5}155\\ \uparrow\ 0 & \uparrow\ 0 & \uparrow\ 0 & \uparrow\ 0 & \uparrow\\ \uparrow\ \uparrow & \uparrow\ \uparrow & \uparrow\ \uparrow & \uparrow\ \uparrow & \uparrow\ \uparrow\end{matrix}$$

$$\begin{vmatrix}
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 3 & 3 & 2 & 0 & 0 & 0 & 0 & 0\\
1 & 4 & 6 & 4 & 2 & 0 & 0 & 0 & 0\\
1 & 5 & 10 & 10 & 5 & 2 & 0 & 0 & 0\\
1 & 6 & 15 & 20 & 15 & 6 & 2 & 0 & 0\\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 2 & 0\\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 2\\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9
\end{vmatrix}$$

Relation between D(k,0) and patterns of determinants of reciprocals of factorials

$$\mathrm{eb}(k) = \frac{D(k,0)}{(2k)!}$$

$$\mathrm{be}(k) = 2^{2k-1}\mathrm{eb}(k) = 2^{2k-1}\frac{D(k,0)}{(2k)!}$$

$$\mathrm{b}(k) = \frac{2^{2k-1}-1}{2^{2k}-1}\mathrm{eb}(k) = \frac{2^{2k-1}-1}{2^{2k}-1}\frac{D(k,0)}{(2k)!}$$

$$-\delta(2k-1) = \frac{(-4)^{k-1}}{k}D(k,0)$$

$$\delta(2k) = 0$$

$$B_{2k} = (-1)^{k-1}\frac{D(k,0)}{2(2^{2k}-1)}$$ Bernoulli number [9]

Aus Symmetriegruenden fuehren wir noch e(k) ein.

$$\begin{array}{ccccccc} \frac{1}{2!} & \frac{5}{4!} & \frac{61}{6!} & \frac{1385}{8!} & \frac{50521}{10!} & \frac{2702765}{12!} & \frac{199360981}{14!} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

$$e(k=7) = \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \frac{1}{12!} & \frac{1}{14!} \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \frac{1}{12!} \\ 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} \\ 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} \\ 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} \end{vmatrix}$$

$$\begin{array}{ccccccccc} & \frac{1}{2!} & \frac{5}{4!} & \frac{61}{6!} & \frac{1385}{8!} & \frac{50521}{10!} & \frac{2702765}{12!} & & \frac{199360981}{14!} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow \end{array}$$

$$e(k=7)=\begin{vmatrix} \frac{1^1}{2!} & \frac{1^3}{4!} & \frac{1^5}{6!} & \frac{1^7}{8!} & \frac{1^9}{10!} & \frac{1^{11}}{12!} & \frac{1^{13}}{14!} \\ \frac{1}{0!} & \frac{1^1}{2!} & \frac{1^3}{4!} & \frac{1^5}{6!} & \frac{1^7}{8!} & \frac{1^9}{10!} & \frac{1^{11}}{12!} \\ 0 & \frac{1}{0!} & \frac{1^1}{2!} & \frac{1^3}{4!} & \frac{1^5}{6!} & \frac{1^7}{8!} & \frac{1^9}{10!} \\ 0 & 0 & \frac{1}{0!} & \frac{1^1}{2!} & \frac{1^3}{4!} & \frac{1^5}{6!} & \frac{1^7}{8!} \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{1^1}{2!} & \frac{1^3}{4!} & \frac{1^5}{6!} \\ 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1^1}{2!} & \frac{1^3}{4!} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1^1}{2!} \end{vmatrix}$$

Relation between e(k) and Euler numbers [10]

$$E_{2k} = (-1)^k (2k)!\, e(k)$$

$$\frac{2}{2!}\quad \frac{16}{4!}\quad \frac{272}{6!}\quad \frac{7936}{8!}\quad \frac{353792}{10!}\quad \frac{22368256}{12!}\quad \frac{1903757312}{14!}$$

$$e^{\#}(k=7) = \begin{vmatrix} \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \frac{2^9}{10!} & \frac{2^{11}}{12!} & \frac{2^{13}}{14!} \\ \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \frac{2^9}{10!} & \frac{2^{11}}{12!} \\ 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \frac{2^9}{10!} \\ 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} \\ 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} \end{vmatrix}$$

$$e^{\#}(k) = (2k+1)be(k+1)$$

$$= (-1)^{k-1}(2k)!\,\delta(2k+1)$$

How about

$$\begin{array}{cccccccc} & \overset{3}{\downarrow} & & \overset{27}{\downarrow} & \overset{243}{\downarrow} & & & \\ \frac{}{2!} & & \frac{}{4!} & & \frac{}{6!} & \frac{}{8!} & \frac{}{10!} & \frac{}{12!} & \frac{}{14!} \end{array}$$

$$e^{\#\#}(k=7)=\begin{vmatrix} \frac{3^1}{2!} & \frac{3^3}{4!} & \frac{3^5}{6!} & \frac{3^7}{8!} & \frac{3^9}{10!} & \frac{3^{11}}{12!} & \frac{3^{13}}{14!} \\ \frac{1}{0!} & \frac{3^1}{2!} & \frac{3^3}{4!} & \frac{3^5}{6!} & \frac{3^7}{8!} & \frac{3^9}{10!} & \frac{3^{11}}{12!} \\ 0 & \frac{1}{0!} & \frac{3^1}{2!} & \frac{3^3}{4!} & \frac{3^5}{6!} & \frac{3^7}{8!} & \frac{3^9}{10!} \\ 0 & 0 & \frac{1}{0!} & \frac{3^1}{2!} & \frac{3^3}{4!} & \frac{3^5}{6!} & \frac{3^7}{8!} \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{3^1}{2!} & \frac{3^3}{4!} & \frac{3^5}{6!} \\ 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{3^1}{2!} & \frac{3^3}{4!} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{3^1}{2!} \end{vmatrix}$$

???

$$\frac{a}{2!}\quad\frac{a^2(6-a)}{4!}$$

$$\uparrow\qquad\uparrow$$

$$e^{\#\#}(k=7,a)=\begin{vmatrix}
\frac{a^1}{2!} & \frac{a^3}{4!} & \frac{a^5}{6!} & \frac{a^7}{8!} & \frac{a^9}{10!} & \frac{a^{11}}{12!} & \frac{a^{13}}{14!} \\
\frac{1}{0!} & \frac{a^1}{2!} & \frac{a^3}{4!} & \frac{a^5}{6!} & \frac{a^7}{8!} & \frac{a^9}{10!} & \frac{a^{11}}{12!} \\
0 & \frac{1}{0!} & \frac{a^1}{2!} & \frac{a^3}{4!} & \frac{a^5}{6!} & \frac{a^7}{8!} & \frac{a^9}{10!} \\
0 & 0 & \frac{1}{0!} & \frac{a^1}{2!} & \frac{a^3}{4!} & \frac{a^5}{6!} & \frac{a^7}{8!} \\
0 & 0 & 0 & \frac{1}{0!} & \frac{a^1}{2!} & \frac{a^3}{4!} & \frac{a^5}{6!} \\
0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{a^1}{2!} & \frac{a^3}{4!} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \frac{a^1}{2!}
\end{vmatrix}$$

$$\lim_{k\to\infty}\frac{e^{**}(k,a)}{e^{**}(k+1,a)}=A\pi^2$$

| a | A |
|---|---|
| $\frac{1}{8}=0.125$ | |
| $\frac{1}{4}=0.25$ | |
| $\frac{1}{3}=0.\bar{3}$ | |
| $\frac{1}{2}=0.5$ | $\frac{4}{9}=0.\bar{4}$ |
| 0.761593314377... | 0.309032344... |
| 1 | $\frac{1}{4}=0.25$ |
| $\frac{13}{11}=1.\overline{18}$ | 0.223087675... |
| $\frac{7}{5}=1.4$ | 0.203136851... |
| $\frac{3}{2}=1.5$ | $\frac{16}{81}=0.197530864...$ |
| $\frac{\pi}{2}=1.570796327...$ | 0.194842905... |
| $\frac{5}{3}=1.\bar{6}$ | 0.19304399... |
| 2 | |

$$\lim_{k\to\infty}[f(k)=\frac{be(k)}{e(k)}]=\frac{\pi}{2}$$

**Convergence behavior of f(k)**

| k | $f(k)=\frac{be(k)}{e(k)}\to\frac{\pi}{2}$ |
|---|---|
| **1** | **1.0** |
| 2 | **1.6** |
| 3 | **1.** 5737704918032786885… |
| 4 | 1.5711191335740072202… |
| 5 | 1.5708319312761030067… |
| 6 | 1.5708002730537061121… |
| 7 | 1.5707967648895146639… |
| 8 | 1.5707963754571858841… |
| 9 | 1.5707963322012269980… |
| 10 | 1.5707963273955764678… |
| 11 | 1.5707963268616378856… |
| 12 | 1.5707963268023122780… |
| 13 | 1.5707963267957205798… |
| 14 | 1.5707963267949881703… |
| 15 | 1.5707963267949067916… |
| 16 | 1.5707963267948977495… |
| 17 | 1.5707963267948967448… |
| 18 | 1.5707963267948966332… |
| 19 | 1.5707963267948966208… |
| 20 | 1.5707963267948966194… |

21 1.57079632679489661923…

22 1.57079632679489661923… ←

23 1.57079632679489661923…

$\frac{\pi}{2} = 1,57079632679489661923,,,$

Difference quotient

$$df = \lim_{k\to\infty} df(k) = \frac{f(k)-f(k+1)}{f(k+1)-f(k+2)} = 9$$

Linear convergence acceleration

$$f1(k) = \frac{1}{df-1}[(df)f(k+1)-f(k)]$$

$$f2(k) = \frac{1}{(df-1)[(df)^2-1]}[(df)^3 f(k+2) - df(df+1)f(k+1)+f(k)]$$

| k | f1(k) |
|---|---|
| 1 | 1.675 |
| 2 | 1.57049180327868852459… |
| 3 | 1.57078771379534828674… |
| 4 | 1.57079603098886498008… |
| 5 | 1.57079631577590650022… |
| 6 | 1.57079632636899073286… |

| | |
|---|---|
| 7 | 1.570796326778144786 7… |
| 8 | 1.570796326794232137 2… |
| 9 | 1.570796326794870151 6… |
| 10 | 1.570796326794895562 8… |
| 11 | 1.570796326794896577 0… |
| 12 | 1.570796326794896617 5… |
| 13 | 1.570796326794896619 2… |
| 14 | 1.570796326794896619 2… |

| k | f2(k) |
|---|---|
| 1 | 1.569185450819672131 1… |
| 2 | 1.570791412676806533 7… |
| 3 | 1.570796134953783938 6… |
| 4 | 1.570796319335744519 3… |
| 5 | 1.570796326501404285 7… |
| 6 | 1.570796326783259212 4… |
| 7 | 1.570796326794433229 1… |
| 8 | 1.570796326794878126 7… |
| 9 | 1.570796326794895880 4… |
| 10 | 1.570796326794896589 7… |
| 11 | 1.570796326794896618 1… |
| 12 | 1.570796326794896619 2… |
| 13 | 1.570796326794896619 2… |

More

$$\lim_{k\to\infty}\frac{e(k)}{e(k+1)}=\lim_{k\to\infty}\frac{be(k)}{be(k+1)}=\left(\frac{\pi}{2}\right)^2$$

$$\lim_{k\to\infty}\frac{eb(k)}{eb(k+1)}=\lim_{k\to\infty}\frac{b(k)}{b(k+1)}=\pi^2$$

Convergence behavior of e(k+1)/e(k) and be(k+1)/be(k) $\rightarrow (2/\pi)^2$

| k | f(k)=e(k+1)/e(k) | g(k)= be(k+1)/be(k) |
|---|---|---|
| 1 | 0.41666666666666666667 | 0.66666666666666666667 |
| 2 | 0.40666666666666666667 | 0.40000000000000000000 |
| 3 | 0.40544496487119437939 | 0.40476190476190476190 |
| 4 | 0.40530284797432811873 | 0.40522875816993464052 |
| 5 | 0.40528676044105271555 | 0.40527859237536656891 |
| 6 | 0.40528496021463408565 | 0.40528405506697836654 |
| 7 | 0.40528475966334321626 | 0.40528465918504373000 |
| 8 | 0.40528473735847146543 | 0.40528472619790335470 |
| 9 | 0.40528473487928945981 | 0.40528473363937263475 |
| 10 | 0.40528473460379013030 | 0.40528473446602742177 |
| 11 | 0.40528473457317770424 | 0.40528473455787096923 |
| 12 | 0.40528473456977626792 | 0.40528473456807552888 |
| 13 | 0.40528473456939832833 | 0.40528473456920935770 |
| 14 | 0.40528473456935633495 | 0.40528473456933533823 |
| 15 | 0.40528473456935166902 | 0.40528473456934933605 |
| 16 | 0.40528473456935115058 | 0.40528473456935089136 |
| 17 | 0.40528473456935109298 | 0.40528473456935106417 |
| 18 | 0.40528473456935108658 | 0.40528473456935108338 |
| 19 | 0.40528473456935108586 | 0.40528473456935108551 |
| 20 | 0.40528473456935108579 | 0.40528473456935108575 |
| 21 | 0.40528473456935108578 | 0.40528473456935108577 |
| 22 | 0.40528473456935108578 | 0.40528473456935108578 |
| 23 | 0.40528473456935108578 | 0.40528473456935108578 |

Difference quotients

$$df = \lim_{k\to\infty}[df(k) = \frac{f(k)-f(k+1)}{f(k+1)-f(k+2)}] = 9$$

$$dg = \lim_{k\to\infty}[dg(k) = \frac{g(k)-g(k+1)}{g(k+1)-g(k+2)}] = 9$$

$$dfg = \lim_{k\to\infty}[dfg(k) = \frac{f(k)-f(k+1)}{g(k+1)-g(k)}] = 3$$

Huygens convergence acceleration

$$z(k) = \frac{1}{dfg+1}[(dfg)f(k)+ g(k)]$$

e.g.

$$z(12) = \frac{1}{4}(3\bullet\ 0,405284734569776267 92$$

$$+\ 0,405284734568075528 88)$$

$$= \quad 0,405284734569351083 16$$

# Math Ship Three

Pascal's triangle, the sums of powers of integers, and Pi squared

$$\sum_{i=1}^{n} i^{m=6} = \frac{\begin{vmatrix} (-n)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$\Delta(k=7,n)=\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 3 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 4 & 4 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 10 & 5 & 0 & 0 \\ p^4 & 0 & 0 & 6 & 20 & 6 & 0 \\ p^5 & 0 & 0 & 1 & 21 & 35 & 7 \\ p^6 & 0 & 0 & 0 & 8 & 56 & 56 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^3} = \frac{\Delta(k,n)}{\Delta(k,1)}$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i^1\right)^2 = \frac{p^2}{4} \; ; \; p = n(n+1)$$

$$\lim_{k\to\infty} \binom{2k+4}{3} \frac{\Delta(k,0)}{\Delta(k+1,0)} = \frac{4}{3}\pi^2$$

# Math Ship Four

Pascal’s triangle and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 1 | | | | | | | | |
| 1 | 2 | 1 | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |

Pascal'striangle

$$dr(k=7)=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 15 & 15 & 1 & 0 \\ 0 & 0 & 0 & 7 & 35 & 21 & 0 \\ 0 & 0 & 0 & 1 & 28 & 70 & 28 \end{vmatrix}$$

Recurrence relation

$$dr(k) = \sum_{i=1}^{k-1} (-1)^{i-1} \binom{k+1}{k+1-2i} dr(k-i)$$

$$\sum_{i=0}^{k} (-1)^{i} \binom{k+1}{k+1-2i} dr(k-i) = 0$$

$$dr(0) = 0,\ dr(1) = 1$$

$$\lim_{k\to\infty} \binom{2k+4}{2} \frac{dr(k)}{dr(k+1)} = \frac{\pi^2}{2}$$

**Convergence behavior**

$$k \quad f(k) = \binom{2k+4}{2} \frac{dr(k)}{dr(k+1)} \to \frac{\pi^2}{2}$$

| | |
|---|---|
| 1 | 5.0 |
| 2 | 4.9411764705882352941.. |
| 3 | 4.9354838709677419355.. |
| 4 | 4.9348769898697539797.. |
| 5 | 4.9348031184344572592.. |
| 6 | 4.9348023024765706802.. |
| 7 | 4.9348022118682237848.. |
| 8 | 4.9348022018024623338.. |
| 9 | 4.9348022006844627741.. |
| 10 | 4.9348022005602106641.. |
| 11 | 4.9348022005464050098.. |
| 12 | 4.9348022005448710537.. |
| 13 | 4.9348022005447006143.. |
| 14 | 4.9348022005446816766.. |
| 15 | 4.9348022005446795724.. |
| 16 | 4.9348022005446793386.. |
| 17 | 4.9348022005446793098.. |
| 18 | 4.9348022005446793095.. |
| 19 | 4.9348022005446793094.. |
| 20 | 4.9348022005446793094.. |

$$\frac{\pi^2}{2} = 4.9348022005446793094..$$

Difference quotient

$$df = \lim_{k\to\infty} df(k) = \frac{f(k)-f(k+1)}{f(k+1)-f(k+2)} = 9$$

Linear convergence acceleration

$$f1(k) = \frac{df}{df-1} f(k+1) - \frac{1}{df-1} f(k)$$

e.g.

$$f1(6) = \frac{9}{8} f(7) - \frac{1}{8} f(6)$$

$$= \frac{9}{8} 4.934802211868223784 8.. -$$

$$\frac{1}{8} 4.934802302476570680 2..$$

$$= \quad 4.934802200542180338 0..$$

Tripel accelaration

$$k \qquad f1(k) = \frac{\begin{vmatrix} f(k) & f(k+1) \\ f(k+1) & f(k+2) \end{vmatrix}}{f(k)-2f(k+1)+f(k+2)} \to \frac{\pi^2}{2}$$

| | |
|---|---|
| 1 | 4.934873949579831932 8 |
| 2 | 4.934804570473071831 7 |
| 3 | 4.934802286620000067 8 |
| 4 | 4.934802203820408730 8 |
| 5 | 4.934802200672363823 1 |
| 6 | 4.934802200549718598 1 |
| 7 | 4.934802200544879484 7 |
| 8 | 4.934802200544687287 7 |
| 9 | 4.934802200544679628 0 |
| 10 | 4.934802200544679322 1 |
| 11 | 4.934802200544679309 9 |
| 12 | 4.934802200544679309 4 |
| 13 | 4.934802200544679309 4 |

Exercise

---

Integration of

$$dr(k=7)=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 15 & 15 & 1 & 0 \\ 0 & 0 & 0 & 7 & 35 & 21 & 0 \\ 0 & 0 & 0 & 1 & 28 & 70 & 28 \end{vmatrix}$$

&

Recurrence relation

$$dr(k)=\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k+1}{k+1-2i}dr(k-i)$$

$$\sum_{i=0}^{k}(-1)^{i}\binom{k+1}{k+1-2i}dr(k-i)=0$$

$$dr(0)=0,\ dr(1)=1$$

&

$$\lim_{k\to\infty}\binom{2k+4}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{2}$$

&

Difference quotient

$$df=\lim_{k\to\infty}df(k)=\frac{f(k)-f(k+1)}{f(k+1)-f(k+2)}=9$$

&

Linear convergence acceleration

$$f1(k)=\frac{df}{df-1}f(k+1)-\frac{1}{df-1}f(k)$$

Factorization of dr(k)

dr(1)=1

dr(2)=3

dr(3)=17

dr(4)=155=5*31

dr(5)=2073=3*691

dr(6)=38227=7*43*127

dr(7)=929569=257*3617

dr(8)=28820619=$3^2$*73*43867

dr(9)=1109652905=5*31*41*283*617

dr(10)=51943281731

= 11*89*131*593*683

dr(11)=2905151042481

=3*17*103*241*2 294797

dr(12)=191329672483963

=13*2731*8191*657931

dr(13)=14655626154768697

=7*43*113*127*9349*362903

dr(14)=129188508448017715

= 3*5*151*331*1721*1001259881

dr(15)=1298481636811073019 53

=37*257*683*65537*305065927

dr(16)=1476144667337841640013 87

=7*43*113*127*9349*362903

dr(17)=1884515541728818675112649

$=3^2*73*109*263152715530534773373$

dr(18)=268463531464165471482681379

=19*174763*524287*154210205991661

dr(19)=4243362672549192531319507118 5=5

*17*31*61681*137616929*

1897170 067619

dr(43)=

1321826542020030568297044907917548857

8441372369270527631439318392777237582

7799226116573307 3=

11*17*307*353*397*683*2113*

2682679*2931542417*

1447585356453146010512453675930977703

5388876684623371934640 9

dr(44)=

1072771528756695855121403860094670387

7155524594108495518119365379361759439

2181427738687651583 95=

$3^2$*5*73*151*331*587*631*23311*

18837001*175831791043 9*

2284701139527905718158077543648209000

7119816174220479539 9

$$db(k=7)=\begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 20 & 6 & 0 & 0 \\ 0 & 0 & 1 & 21 & 35 & 7 & 0 \\ 0 & 0 & 0 & 8 & 56 & 56 & 8 \\ 0 & 0 & 0 & 1 & 36 & 126 & 84 \end{vmatrix}$$

Factorization of db(k)

db(1)=1

db(2)=4=$2^2$

db(3)=36=$2^2*3^2$

db(4)=600=$2^3*3*5^2$

db(5)=16584=$2^3*3$*691

db(6)=705600=$2^6*3^2*5^2*7^2$

db(7)=43751232=$2^6*3^3*7$*3617

db(8)=3790108800=$2^7*3^3*5^2$*43867

db(9)=4435398777 60

=$2^7*3^4*5*7^2$*283*617

db(10)=6821884903680 0

=$2^9*3^4*5^2*7*11^2$*131*593

db(11)=1347842592518400 0

=$2^9*3^4*5^3*11$*103*2294797

db(12)=3355402067989171200

=$2^{10}*3^7*5^2*7^2*11*13^2$*657931

db(13)=10352187147146068224 00
$=2^{10}*3^5*5^2*7^3*11*13*9349*362903$
db(14)=3901892569831394611 20000
$=2^{14}*3^5*5^4*7*13*1721*1001259881$
db(15)=
177430554 75697274669506560 0 =
$2^{14}*3^6*5^2*7^2*11^2*13*37*683*305065927$
db(16)=
962693723015686771703193 60000=
$2^{15}*3^7*5^4*7^3*11*13*17^2*151628697551$
db(17)=
617143599451874042792419811328 00=$2^{15}*3^7*5^2*7*11*17*$
263152715530534773 73
db(18)=
463345225000480030642943923814 40000=$2^{17}3^8*5^4*7^2*11*13^2*17*19^2*$
15421020599166 1
db(19)=
404210971901053839010954023571783680 00=$2^{17}*3^8*5^3*7^3*13*17*19*$
137616929*1897170067619
db(20)=
406807347156413884029227310312687206 40000=$2^{18}*3^8*5^4*7^2*11^2*13*$

17*19*1520097643918070802691

db(21)=
4692692289534704936457797450811151466
4960000=$2^{18}*3^{10}*5^{4}*7^{3}*11^{3}*13*17*$
19*59*8089*2947939*1798482437

db(43)=
2273561580826151930385386634787006060
5265645635372206346884819164296880655
5806293206548949
4935624194981561683148800000000=$2^{40}*3$
$^{20}*5^{9}*7^{6}*11^{5}*13^{3}*17^{3}*19^{2}*29*31*37*$
41*43*307*2682679*
1447585356453146010512453675930977703
5388876684623371934640 9

db(44)=
2169645193342054344666010266722899331
4838387915077313702922342485590403411
3542716417644677279431176422816383049
4003200000000000=
$2^{41}*3^{20}*5^{11}*7^{6}*11^{3}*13^{4}*17^{2}*19*23^{2}*29*37*$
41*43*587*175817910439*

228470113952790571815807754364820900071198161742204795399

db(45)=
221002718404986554298240768755333189429920146405352431762408283414413605871697927426749032291216799875329288779753817374720000000000=$2^{41}*3^{21}*5^{9}*7^{6}*11^{4}*13^{3}*17^{2}*19^{2}*23^{3}*29*31^{2}*37*41*43$*587*108023*88834989941192452064696371697041093440592668865837981613684998 9

db(46)=
239955426193823777192083054923229795250300982818383729132766938890060944970167958203196527465667439702450909210662945580646400000000000=$2^{45}*3^{21}*5^{11}*7^{6}*11^{4}*13^{3}*17^{2}*19^{3}*23^{2}*29*31*37*41*43*47$*467*1499*2459153*421712661774158957599564 1*357792201382727497686063184090028 9

db(47)=
277338323446486436498507683627119669714834413172383174930342429634320585 92

589870965569387341865090752788594287136632039762831605760000000000=
$2^{45}*3^{21}*5^{9}*7^{7}*11^{4}*13^{2}*17*19^{2}*23^{2}*29*31*37*41*43*47$*7823741903*
4155593423131*1001795243652611 3*
96454277809515481*
673548016777364487369127 1

db(48)=
340785747929879171950628513333258486976729107021295547196588019324192603702577754405604673690295917180871388371830792159186617630720000000000000=
$2^{46}*3^{23}*5^{12}*7^{10}*11^{5}*13^{3}*17^{2}*19^{2}*23^{2}*29*31*37*41*43*47$*2857*3221*1671211*
92157896932766071 67*97782631528749962185846173071805496164355 99

db(60)=
320203879587530537257338060379565151072136029461521227281464669859549798296590286505229909031212888362273078685591942913134631732972397821683482678773126812564214681910237416390656000000000000000=

$2^{56}*3^{28}*5^{14}*7^{9}*11^{5}*13^{4}*17^{3}*19^{3}*23^{2}*29^{2}*31^{2}*37*41^{2}*43*47*53*59*61^{2}$*154531458643314256044752653787402290496715484256534340547945511741230258373473403400074026079180721100589360118083714596963007831919923124910245113526428180585303261250502875262061874081441783419902893290728587672701147161142035482206031365901788986432575738953802532819155766580787095140927481881792265351109949433175965007640044676727201275132563751245661139767708267584627688416062076079239070320651248618344848491859148800000000000000000000000 = $2^{96}*3^{48}*5^{23}*7^{16}*11^{8}*13^{7}*17^{5}*19^{5}*23^{4}*29^{3}*31^{3}*37^{2}*41*43^{2}*47^{2}*53*59*61*67^{2}*71*73*79*83*89*97$*…

db(100)=

690980911406176284863700682866320936060118083714596963007831919923124910245113526428180585303261250502875262061874081441783419902893290728587672701147161142035482206031365901788986432575738953802532819155766580787095140927481881792265351109949433175965007640044676727201275132563751245661139767708267584627688416062076079239070320651248618344848491859148800000000000000000000000 =
$2^{96}*3^{48}*5^{23}*7^{16}*11^{8}*13^{7}*17^{5}*19^{5}*23^{4}*29^{3}*31^{3}*37^{2}*41*43^{2}*47^{2}*53*59*61*67^{2}*71*73*79*83*89*97$*…

db(91)=
16226746127521460151409283603640746343786746329896355613071597837529649822921624294812470022983719756381195576138620550706119850283791267442349241719935420849335004400716740204119658595937370688684705839173067798247518540652039482102497078426475786949508284107878471351858712399957819731050601930651507650409267200000000000000000000 =
$2^{87}*3^{44}*5^{21}*7^{15}*11^{8}*13^{8}*17^{5}*19^{4}*23^{4}*29^{3}*31^{2}*37^{2}*41^{2}*43^{2}*47*53*59*61^{2}*67*71*73*79*83*89$*731071444752614233110898416186 633327*362702761564874666647 7*2122174114227419648093461601*268502092368106529218945285817057918976311791467905416291021095148406738207575946053456408281673680 375113

The prime factorization structures of dr(k) and db(k) are significantly different!

db(k) contains many small primaries (small "noise"), more precisely the mostly complete sequence of primaries from 2 to about k, the order of the prime factor 2 is about k , of 3 about int(k/2), of 5 about int(k/4) and so on to 1(the larger k, the smaller is the "about"), and a small number of increasingly large primes ("factor bombs"), comparable with "monster waves" in special simple continued fractions (see Unit 30).

Recurrence relation
??????????????????

$$\lim_{k\to\infty}\binom{2k+6}{3}\frac{db(k)}{db(k+1)}=\frac{4\pi^2}{3}$$

The quotient of two immediately consecutive $\begin{cases}\text{row sums}\\ \text{column sums}\end{cases}$ of the determinants dr(k) and

$db(k)$ converges to $\begin{cases}\Phi^2=\frac{1}{2}(3+\sqrt{5})\\ 2\end{cases}$ for k to $\infty$.

# Math Ship Five

Modified Pascal’s triangle and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 2 | | | | | | | | |
| 1 | 3 | 2 | | | | | | | |
| 1 | 4 | 5 | 2 | | | | | | |
| 1 | 5 | 9 | 7 | 2 | | | | | |
| 1 | 6 | 14 | 16 | 9 | 2 | | | | |
| 1 | 7 | 20 | 30 | 25 | 11 | 2 | | | |
| 1 | 8 | 27 | 50 | 55 | 36 | 13 | 2 | | |
| 1 | 9 | 35 | 77 | 105 | 91 | 49 | 15 | 2 | |
| 1 | 10 | 44 | 112 | 182 | 196 | 140 | 64 | 17 | 2 |

Modified Pascal's triangle

$$dr(k=7)=\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 9 & 2 & 0 & 0 & 0 \\ 0 & 0 & 6 & 16 & 2 & 0 & 0 \\ 0 & 0 & 1 & 20 & 25 & 2 & 0 \\ 0 & 0 & 0 & 8 & 50 & 36 & 2 \\ 0 & 0 & 0 & 1 & 35 & 105 & 49 \end{vmatrix}$$

Factorization of dr(k)

dr(16)= 569062454791340571761704 96 = $2^{16}$*43691*131071*151628697551 dr(17)= 13722623393637762299131396096= $2^{16}$*73*109*2631527155305347737 3 dr(18)= 37040054732706417555976856535 04= $2^{18}$*174763*524287*154210205991661 dr(19)= 11123720644327355269302208740720 64=$2^{17}$ *17*31*61681*137616929* 1897170 067619

$$db(k=7)=\begin{vmatrix} 1 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 14 & 9 & 0 & 0 & 0 \\ 0 & 0 & 7 & 30 & 11 & 0 & 0 \\ 0 & 0 & 1 & 27 & 55 & 13 & 0 \\ 0 & 0 & 0 & 9 & 77 & 91 & 15 \\ 0 & 0 & 0 & 1 & 44 & 182 & 140 \end{vmatrix}$$

Factorization of db(k)
db(16)=
19044223870038891330450207000937 5 =
$3^8*5^5*7^3*11^2*13*17^2*19*23*29*31*$
151628697551

db(17)=
22489293407860857419060956461864562 5=
$3^8*5^4*7^2*11^2*17*23*29*31*$
263152715530534773 73
The prime factorization structures of dr(k) and db(k) are significantly different!
db(k) contains many small primaries (small "noise") and a small number of large primes ("factor bombs"), comparable with "monster waves" in special simple continued fractions (see
Unit 30).

$$\lim_{k\to\infty}\binom{2k+3}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{4}$$

$$\lim_{k\to\infty}\binom{2k+5}{3}\frac{db(k)}{db(k+1)}=\frac{2\pi^2}{3}$$

The quotient of two immediately consecutive $\begin{cases} \text{row sums} \\ \text{column sums} \end{cases}$ of the determinants dr(k) and db(k) converges to $\begin{cases} \Phi^2 = \frac{1}{2}(3+\sqrt{5}) \\ 2 \end{cases}$ for k to $\infty$.

# Math Ship Six

Modified Pascal's triangle and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 3 | | | | | | | | |
| 1 | 4 | 3 | | | | | | | |
| 1 | 5 | 7 | 3 | | | | | | |
| 1 | 6 | 12 | 10 | 3 | | | | | |
| 1 | 7 | 18 | 22 | 13 | 3 | | | | |
| 1 | 8 | 25 | 40 | 35 | 16 | 3 | | | |
| 1 | 9 | 33 | 65 | 75 | 51 | 19 | 3 | | |
| 1 | 10 | 42 | 98 | 140 | 126 | 70 | 22 | 3 | |
| 1 | 11 | 52 | 140 | 238 | 266 | 196 | 92 | 25 | 3 |

Modified Pascal'striangle

$$dr(k=7)=\begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 12 & 3 & 0 & 0 & 0 \\ 0 & 0 & 7 & 22 & 3 & 0 & 0 \\ 0 & 0 & 1 & 25 & 35 & 3 & 0 \\ 0 & 0 & 0 & 9 & 65 & 51 & 3 \\ 0 & 0 & 0 & 1 & 42 & 140 & 70 \end{vmatrix}$$

$$db(k=7)=\begin{vmatrix} 1 & 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 10 & 0 & 0 & 0 & 0 \\ 0 & 1 & 18 & 13 & 0 & 0 & 0 \\ 0 & 0 & 8 & 40 & 16 & 0 & 0 \\ 0 & 0 & 1 & 33 & 75 & 19 & 0 \\ 0 & 0 & 0 & 10 & 98 & 126 & 22 \\ 0 & 0 & 0 & 1 & 52 & 238 & 196 \end{vmatrix}$$

$$\lim_{k\to\infty}\binom{2k+4}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{2}$$

$$\lim_{k\to\infty}\binom{2k+6}{3}\frac{db(k)}{db(k+1)}=\frac{4\pi^2}{3}$$

The quotient of two immediately consecutive

$$\begin{cases} \text{row sums} \\ \text{column sums} \end{cases}$$ of the determinants dr(k) and

db(k) converges to $\begin{cases} \Phi^2 = \frac{1}{2}(3+\sqrt{5}) \\ 2 \end{cases}$ for k to $\infty$.

# Math Ship Seven

Modified Pascal's triangle and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 3 | | | | | | | | |
| 1 | 4 | 4 | | | | | | | |
| 1 | 5 | 8 | 4 | | | | | | |
| 1 | 6 | 13 | 12 | 4 | | | | | |
| 1 | 7 | 19 | 25 | 16 | 4 | | | | |
| 1 | 8 | 26 | 44 | 41 | 20 | 4 | | | |
| 1 | 9 | 34 | 70 | 85 | 61 | 24 | 4 | | |
| 1 | 10 | 43 | 104 | 155 | 146 | 85 | 28 | 4 | |
| 1 | 11 | 53 | 147 | 259 | 301 | 231 | 113 | 32 | 4 |

Modified Pascal'striangle

$$dr(k=7)=\begin{vmatrix} 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 13 & 4 & 0 & 0 & 0 \\ 0 & 0 & 7 & 25 & 4 & 0 & 0 \\ 0 & 0 & 1 & 26 & 41 & 4 & 0 \\ 0 & 0 & 0 & 9 & 70 & 61 & 4 \\ 0 & 0 & 0 & 1 & 43 & 155 & 85 \end{vmatrix}$$

$$db(k=7)=\begin{vmatrix} 1 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 12 & 0 & 0 & 0 & 0 \\ 0 & 1 & 19 & 16 & 0 & 0 & 0 \\ 0 & 0 & 8 & 44 & 20 & 0 & 0 \\ 0 & 0 & 1 & 34 & 85 & 24 & 0 \\ 0 & 0 & 0 & 10 & 104 & 146 & 28 \\ 0 & 0 & 0 & 1 & 53 & 259 & 231 \end{vmatrix}$$

Factorization of db(k)

db(1)=1

db(2)=6=2*3

db(3)=102=2*3*17

db(4)=3720=$2^3$*3*5*31

$db(5)=248760=2^3*3^2*5*691$
$db(6)=27523440=2^4*3^2*5*7*43*127$
$db(7)=4685027760=2^4*3^2*5*7*257*3617$

$$\lim_{k\to\infty}\binom{2k+2}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{8}$$

$$\lim_{k\to\infty}\binom{2k+4}{3}\frac{db(k)}{db(k+1)}=\frac{\pi^2}{3}$$

The quotient of two immediately consecutive $\begin{cases}\text{row sums}\\ \text{column sums}\end{cases}$ of the determinants $dr(k)$ and

$db(k)$ converges to $\begin{cases}\Phi^2=\frac{1}{2}(3+\sqrt{5})\\ 2\end{cases}$ for k to $\infty$.

# Math Ship Eight

Modified Pascal’s triangle, the sum of powers of integers, and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 2 | | | | | | | | |
| 1 | 3 | 2 | | | | | | | |
| 1 | 4 | 5 | 2 | | | | | | |
| 1 | 5 | 9 | 7 | 2 | | | | | |
| 1 | 6 | 14 | 16 | 9 | 2 | | | | |
| 1 | 7 | 20 | 30 | 25 | 11 | 2 | | | |
| 1 | 8 | 27 | 50 | 55 | 36 | 13 | 2 | | |
| 1 | 9 | 35 | 77 | 105 | 91 | 49 | 15 | 2 | |
| 1 | 10 | 44 | 112 | 182 | 196 | 140 | 64 | 17 | 2 |

Modified Pascal's triangle

$$?(m=6,n)=\frac{\begin{vmatrix} (-n)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 3 & 0 & 0 & 0 & 0 \\ (-n)^3 & 1 & 4 & 5 & 0 & 0 & 0 \\ (-n)^4 & 1 & 5 & 9 & 7 & 0 & 0 \\ (-n)^5 & 1 & 6 & 14 & 16 & 9 & 0 \\ (-n)^6 & 1 & 7 & 20 & 30 & 25 & 11 \\ (-n)^7 & 1 & 8 & 27 & 50 & 35 & 36 \end{vmatrix}}{\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 3 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 4 & 5 & 0 & 0 & 0 \\ (-1)^4 & 1 & 5 & 9 & 7 & 0 & 0 \\ (-1)^5 & 1 & 6 & 14 & 16 & 9 & 0 \\ (-1)^6 & 1 & 7 & 20 & 30 & 25 & 11 \\ (-1)^7 & 1 & 8 & 27 & 50 & 55 & 36 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$\Gamma(k=7,n)=\begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 5 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 5 & 7 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 14 & 9 & 0 & 0 \\ p^4 & 0 & 0 & 7 & 30 & 11 & 0 \\ p^5 & 0 & 0 & 1 & 27 & 55 & 13 \\ p^6 & 0 & 0 & 0 & 9 & 77 & 91 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n} i^{2k}}{\sum_{i=1}^{n} i^{2}} = \frac{\Gamma(k,n)}{\Gamma(k,1)}$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}p(2n+1)\ ;\ p=n(n+1)$$

$$\lim_{k\to\infty}\binom{2k+5}{3}\frac{\Gamma(k,0)}{\Gamma(k+1,0)} = \frac{2\pi^2}{3}$$

$$G(k=7,n)=\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 4 & 2 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 9 & 2 & 0 & 0 \\ p^4 & 0 & 0 & 6 & 16 & 2 & 0 \\ p^5 & 0 & 0 & 1 & 20 & 25 & 2 \\ p^6 & 0 & 0 & 0 & 8 & 50 & 36 \end{vmatrix}$$

7 by 7 determinant

$$\frac{?}{??}=\frac{\Delta(k,n)}{\Delta(k,1)}$$

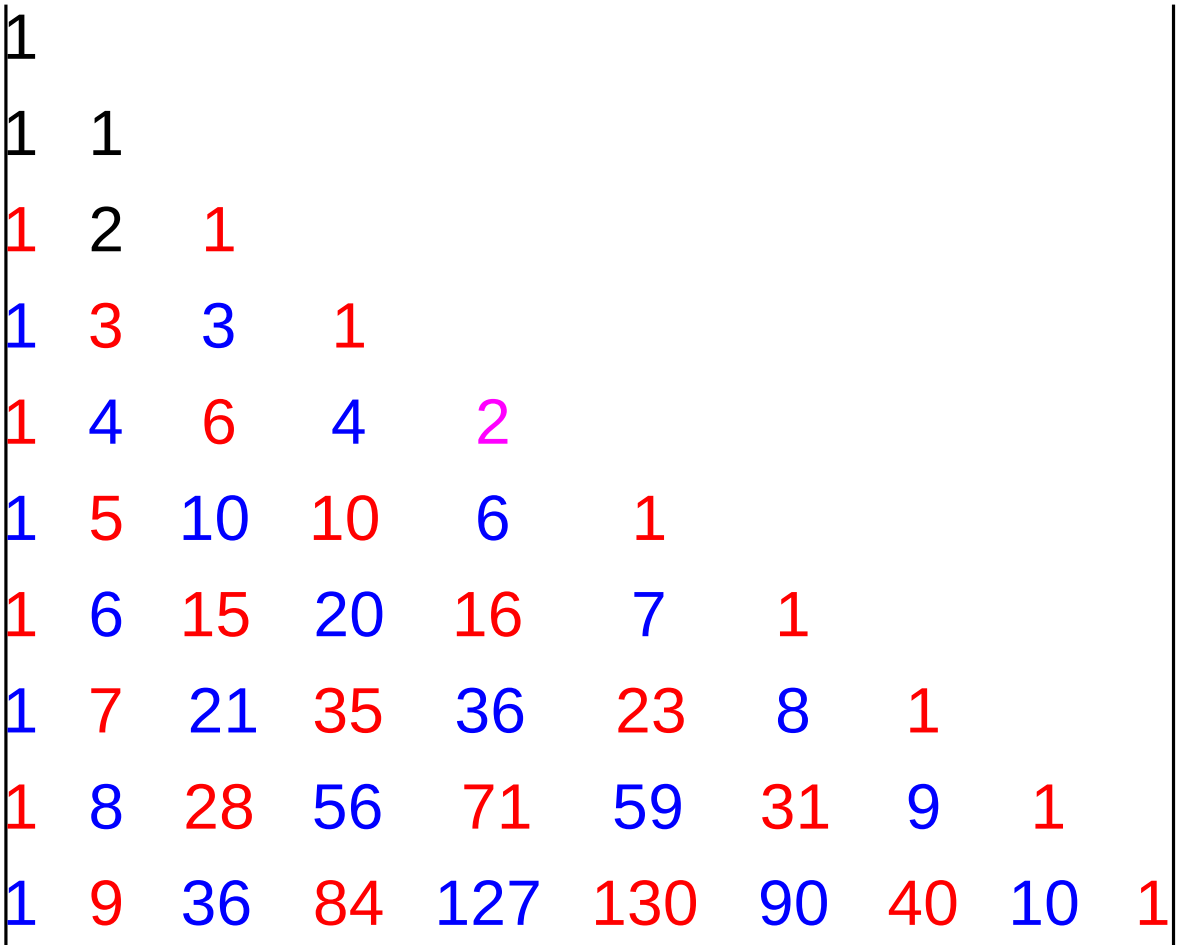
$$\lim_{k\to\infty}\binom{2k+3}{2}\frac{G(k,0)}{G(k+1,0)}=\frac{\pi^2}{3}$$

# Math Ship with error 1

Pascal’s triangle and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 1 | | | | | | | | |
| 1 | 2 | 1 | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |

Pascal'striangle




Pascal's triangle with error

$$dr(k=7)=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 10 & 1 & 0 & 0 \\ 0 & 0 & 1 & 15 & 16 & 1 & 0 \\ 0 & 0 & 0 & 7 & 35 & 23 & 0 \\ 0 & 0 & 0 & 1 & 28 & 71 & 31 \end{vmatrix}$$

$$db(k=7)=\begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 6 & 0 & 0 & 0 \\ 0 & 0 & 6 & 20 & 7 & 0 & 0 \\ 0 & 0 & 1 & 21 & 36 & 8 & 0 \\ 0 & 0 & 0 & 8 & 56 & 59 & 9 \\ 0 & 0 & 0 & 1 & 36 & 127 & 90 \end{vmatrix}$$

$$\lim_{k\to\infty}\binom{2k+4}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{2}$$

$$\lim_{k\to\infty}\binom{2k+6}{3}\frac{db(k)}{db(k+1)}=\frac{4\pi^2}{3}$$

The quotient of two immediately consecutive $\begin{cases} \text{row sums} \\ \text{column sums} \end{cases}$ of the determinants dr(k) and db(k) converges to $\begin{cases} \Phi^2 = \frac{1}{2}(3+\sqrt{5}) \\ 2 \end{cases}$ for k to $\infty$.

# Math Ship with error 2

Pascal's triangle and Pi squared

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 1 | | | | | | | | |
| 1 | 2 | 1 | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |

Pascal'striangle

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 1 | 1 | | | | | | | | |
| 1 | 2 | 1 | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | |
| 1 | 5 | 10 | 37 | 5 | 1 | | | | |
| 1 | 6 | 15 | 47 | 42 | 6 | 1 | | | |
| 1 | 7 | 21 | 62 | 89 | 48 | 7 | 1 | | |
| 1 | 8 | 28 | 83 | 151 | 137 | 55 | 8 | 1 | |
| 1 | 9 | 36 | 111 | 234 | 288 | 192 | 63 | 9 | 1 |

Pascal's triangle with error 2

$$dr(k=7)=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 37 & 1 & 0 & 0 \\ 0 & 0 & 1 & 15 & 42 & 1 & 0 \\ 0 & 0 & 0 & 7 & 62 & 48 & 0 \\ 0 & 0 & 0 & 1 & 28 & 151 & 55 \end{vmatrix}$$

$$db(k=7)=\begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 47 & 6 & 0 & 0 \\ 0 & 0 & 1 & 21 & 89 & 7 & 0 \\ 0 & 0 & 0 & 8 & 83 & 137 & 8 \\ 0 & 0 & 0 & 1 & 36 & 234 & 192 \end{vmatrix}$$

$$\lim_{k\to\infty}\binom{2k+4}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{2}$$

$$\lim_{k\to\infty}\binom{2k+6}{3}\frac{db(k)}{db(k+1)}=\frac{4\pi^2}{3}$$

The quotient of two immediately consecutive $\begin{cases} \text{row sums} \\ \text{column sums} \end{cases}$ of the determinants $dr(k)$ and $db(k)$ converges to $\begin{cases} \Phi^2=\frac{1}{2}(3+\sqrt{5}) \\ 2 \end{cases}$ for k to $\infty$.

# Math Ship Eighteen

Pascal's triangle, the sums of powers of integers, Pi squared, and more

$$\sum_{i=1}^{n} i^6 = \frac{\begin{vmatrix} (-n)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinant

$$\sum_{i=1}^{n}(-1)^{n-i}i^{m=6} = \frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ n^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ n^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ n^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ n^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ n^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

$(6+1)$ by $(6+1)$ determinant

$$= \frac{(n \bmod 2)\delta(m) - d(n,m)}{\delta(m) - d(n=1,m)}$$

$$\delta(m=6) = \begin{vmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ +1 & -2 & 2 & 0 & 0 & 0 \\ -1 & +3 & -3 & 2 & 0 & 0 \\ +1 & -4 & +6 & -4 & 2 & 0 \\ -1 & +5 & -10 & +10 & -5 & 2 \\ +1 & -6 & +15 & -20 & +15 & -6 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 \\ 1 & 4 & 6 & 4 & 2 & 0 \\ 1 & 5 & 10 & 10 & 5 & 2 \\ 1 & 6 & 15 & 20 & 15 & 6 \end{vmatrix}$$

6 by 6 determinants

$$d(n, m=6) = \begin{vmatrix} n^1 & 2 & 0 & 0 & 0 & 0 \\ n^2 & -2 & 2 & 0 & 0 & 0 \\ n^3 & +3 & -3 & 2 & 0 & 0 \\ n^4 & -4 & +6 & -4 & 2 & 0 \\ n^5 & +5 & -10 & +10 & -5 & 2 \\ n^6 & -6 & +15 & -20 & +15 & -6 \end{vmatrix}$$

6 by 6 determinant

$$\delta(m) - d(n=1, m) = (-2)^m$$

Reduction of $\delta(m=2k)$ $\qquad$ $\delta(m=2k) = 0$

Reduction of $\delta(2k-1)$

$\delta(2k-1) =$

$$(-1)^k (2k-1)! \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & 0 & \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \\ \frac{1}{11!} & \frac{1}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

k by k determinant

| k | $\delta(2k-1)$ |
|---|---|
| 1 | –1 |
| 2 | 2 |
| 3 | –16 |
| 4 | 272 |
| 5 | –7936 |
| 6 | 353792 |
| 7 | –22368256 |
| 8 | 1903757312 |

Recurrence relation of $\delta(2k-1)$

$$\sum_{i=1}^{k}\binom{2k+1}{2i}\frac{2i}{2^{2i}(2^{2i}-1)}\delta(2i-1)+(k-\frac{1}{2})=0$$

Connection between $\delta(m=2k-1)$ and the Bernoulli numbers $B_{2k}$

$$\delta(2k-1)=-\frac{2^{2k}(2^{2k}-1)}{2k}B_{2k}$$

$$\lim_{k\to\infty}\binom{2k+1}{2}\frac{\delta(2k-1)}{\delta(2k+1)}=-\frac{\pi^2}{8}$$

Reduction of d(n,m=2k) $d(n, m=2k/k=7)=$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| n | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^2$ | -2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^3$ | 3 | -3 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^4$ | -4 | 6 | -4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^5$ | 5 | -10 | 10 | -5 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^6$ | -6 | 15 | -20 | 15 | -6 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^7$ | 7 | -21 | 35 | -35 | 21 | -7 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n^8$ | -8 | 28 | -56 | 70 | -56 | 28 | -8 | 2 | 0 | 0 | 0 | 0 | 0 |
| $n^9$ | 9 | -36 | 84 | -126 | 126 | -84 | 36 | -9 | 2 | 0 | 0 | 0 | 0 |
| $n^{10}$ | -10 | 45 | -120 | 210 | -252 | 210 | -120 | 45 | -10 | 2 | 0 | 0 | 0 |
| $n^{11}$ | 11 | -55 | 165 | -330 | 462 | -462 | 330 | -165 | 55 | -11 | 2 | 0 | 0 |
| $n^{12}$ | -12 | 66 | -220 | 495 | -792 | 924 | -792 | 495 | -220 | 66 | -12 | 2 | 0 |
| $n^{13}$ | 13 | -78 | 286 | -715 | 1287 | -1716 | 1716 | -1287 | 715 | -286 | 78 | -13 | 2 |
| $n^{14}$ | -14 | 91 | -364 | 1001 | -2002 | 3003 | -3432 | 3003 | -2002 | 1001 | -364 | 91 | -14 |

14 by 14 determinant

$$= (-1)^k 2^{2k-1} \begin{vmatrix} p & 1 & 0 & 0 & 0 & 0 & 0 \\ p^2 & 1 & 1 & 0 & 0 & 0 & 0 \\ p^3 & 0 & 3 & 1 & 0 & 0 & 0 \\ p^4 & 0 & 1 & 6 & 1 & 0 & 0 \\ p^5 & 0 & 0 & 5 & 10 & 1 & 0 \\ p^6 & 0 & 0 & 1 & 15 & 15 & 1 \\ p^7 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant

$$P=n(n+1)$$

$$\text{Reduction of } \sum_{i=1}^{n}(-1)^{n-i} i^m$$

$$\sum_{i=1}^{n}(-1)^{n-i} i^m = -d(n,m)\begin{cases} \dfrac{1}{d(n=1,m)} & m=2k \\ \dfrac{1}{(-2)^m} & n=2j \end{cases} \text{ for}$$

$$\lim_{k\to\infty}\binom{2k+1}{2}\frac{\delta(2k-1)}{\delta(2k+1)} = -\frac{\pi^2}{8}$$

$$\tan x =$$

$$\frac{-\delta(1)}{1!}x^1 + \frac{\delta(3)}{3!}x^3 + \frac{-\delta(5)}{5!}x^5 + \frac{\delta(7)}{7!}x^7 + \frac{-\delta(9)}{9!}x^9 + \dots$$

$$= \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -(2x)^1 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \\ (2x)^3 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \\ -(2x)^5 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \\ (2x)^7 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \\ -(2x)^9 & 0 & 0 & 0 & 0 & \frac{1}{1!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$-\tanh x =$$

$$\frac{\delta(1)}{1!}x^1+\frac{\delta(3)}{3!}x^3+\frac{\delta(5)}{5!}x^5+\frac{\delta(7)}{7!}x^7+\frac{\delta(9)}{9!}x^9+\ldots$$

$$=\begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\tan^2 x =$$

$$\frac{\delta(3)}{2!}x^2 + \frac{-\delta(5)}{4!}x^4 + \frac{\delta(7)}{6!}x^6 + \frac{-\delta(9)}{8!}x^8 + \frac{\delta(11)}{10!}x^{10} + \ldots$$

$$= \begin{vmatrix} 0 & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \frac{2^9}{10!} & \rightarrow \\ -x^2 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \\ x^4 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \\ x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

But

$$\frac{\tan x}{\cos x} = \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \\ x^3 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \\ x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= \frac{1}{1!}x + \frac{5}{3!}x^3 + \frac{61}{5!}x^5 + \frac{1385}{7!}x^7 + \ldots$$

$$\frac{1}{\cos x} = \begin{vmatrix} 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= \frac{1}{0!} + \frac{1}{2!}x^2 + \frac{5}{4!}x^4 + \frac{61}{6!}x^6 + \frac{1385}{8!}x^8 + \dots$$

$$\frac{1}{\cos^2 x} = \begin{vmatrix} 1 & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \frac{2^9}{10!} & \rightarrow \\ -x^2 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \frac{2^7}{8!} & \\ x^4 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \frac{2^5}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \frac{2^3}{4!} & \\ x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{2^1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= \frac{1}{0!} + \frac{2}{2!}x^2 + \frac{16}{4!}x^4 + \frac{272}{6!}x^6 + \frac{7936}{8!}x^8 + \dots$$

$$\frac{1}{\cosh x} = \begin{vmatrix} 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= \frac{1}{0!} - \frac{1}{2!}x^2 + \frac{5}{4!}x^4 - \frac{61}{6!}x^6 + \frac{1385}{8!}x^8 - + \ldots$$

$$\frac{\cosh x}{\cos x} = -\begin{vmatrix} 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \rightarrow \\ 1 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} & \\ x^2 & 0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{-1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & 1 & \frac{1^0}{2!} & \frac{3^0}{4!} & \frac{5^0}{6!} & \frac{7^0}{8!} & \rightarrow \\ 1 & 1 & \frac{-1^0}{2!} & \frac{3^0}{4!} & \frac{-5^0}{6!} & \frac{7^0}{8!} & \\ x^2 & 0 & 1 & \frac{-1^0}{2!} & \frac{3^0}{4!} & \frac{-5^0}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{-1^0}{2!} & \frac{3^0}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{-1^0}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{\cos x}{\cosh x} = \frac{\cos ix}{\cosh ix} = -\begin{vmatrix} 0 & 1 & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} & \rightarrow \\ 1 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^2 & 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1}{2!} & \frac{1}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{(\cosh x)'}{(\cos x)'} = \frac{\cosh x - x\sinh x}{\cos x + x\sin x} =$$

$$= -\begin{vmatrix} 0 & 1 & \frac{-1^1}{2!} & \frac{-3^1}{4!} & \frac{-5^1}{6!} & \frac{-7^1}{8!} & \rightarrow \\ 1 & 1 & \frac{1^1}{2!} & \frac{-3^1}{4!} & \frac{5^1}{6!} & \frac{-7^1}{8!} & \\ x^2 & 0 & 1 & \frac{1^1}{2!} & \frac{-3^1}{4!} & \frac{5^1}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1^1}{2!} & \frac{-3^1}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1^1}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{(\cosh)^{\square\square}}{(\cos x)^{\square\square}} = \frac{\cosh x - x\sinh x + x^2\cosh x}{\cos x + x\sin x - x^2\cos x}$$

$$= - \begin{vmatrix} 0 & 1 & \frac{1^2}{2!} & \frac{3^2}{4!} & \frac{5^2}{6!} & \frac{7^2}{8!} & \rightarrow \\ 1 & 1 & \frac{-1^2}{2!} & \frac{3^2}{4!} & \frac{-5^2}{6!} & \frac{7^2}{8!} & \\ x^2 & 0 & 1 & \frac{-1^2}{2!} & \frac{3^2}{4!} & \frac{-5^2}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{-1^2}{2!} & \frac{3^2}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{-1^2}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{(\cosh)^{\square\square}}{(\cos x)^{\square\square\square}} = \frac{\cosh x - x\sinh x + x^2\cosh x}{\cos x + x\sin x - x^3\sin x}$$

$$= -\begin{vmatrix} 0 & 1 & \frac{1^2}{2!} & \frac{3^2}{4!} & \frac{5^2}{6!} & \frac{7^2}{8!} & \rightarrow \\ x^0 & 1 & \frac{1^3}{2!} & \frac{-3^3}{4!} & \frac{5^3}{6!} & \frac{-7^3}{8!} & \\ x^2 & 0 & 1 & \frac{1^3}{2!} & \frac{-3^3}{4!} & \frac{5^3}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1^3}{2!} & \frac{-3^3}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1^3}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{(\cosh)^{\square\square\square}}{(\cos x)^{\square\square\square}} = \frac{\cosh x - x\sinh x - x^3\sinh x}{\cos x + x\sin x - x^3\sin x}$$

$$= - \begin{vmatrix} 0 & 1 & \frac{-1^3}{2!} & \frac{-3^3}{4!} & \frac{-5^3}{6!} & \frac{-7^3}{8!} & \rightarrow \\ x^0 & 1 & \frac{1^3}{2!} & \frac{-3^3}{4!} & \frac{5^3}{6!} & \frac{-7^3}{8!} & \\ x^2 & 0 & 1 & \frac{1^3}{2!} & \frac{-3^3}{4!} & \frac{5^3}{6!} & \\ x^4 & 0 & 0 & 1 & \frac{1^3}{2!} & \frac{-3^3}{4!} & \\ x^6 & 0 & 0 & 0 & 1 & \frac{1^3}{2!} & \\ x^8 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{x}{\sin x} = \begin{vmatrix} 1 & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \rightarrow \\ -x^2 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \\ +x^4 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \\ -x^6 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{1!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \ldots$$

Connection between $\delta(m= 2k-1)$ and the Bernoulli numbers $B_{2k}$

$$\delta(2k-1) = -\frac{2^{2k}(2^{2k}-1)}{2k} B_{2k}$$

$$\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = -(7!)$$

(6+1) by (6+1) determinant

$$\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix} = (-2)^6$$

(6+1) by (6+1) determinant

**In number theory, 691 is a "marker" (similar to the radioactive markers in biology): whenever it appears in a computation, one can be sure that Bernoulli numbers are involved.**

http://en.wikipedia.org/wiki/691_%28number%29

# Math Ship
# Hundred Thirty Seven
# The general sandbox

** PT – die rote und die blaue Partei
** Determinant patterns of reciprocals of Factorials – the Bernoulli and the Euler Wing

**DPRF Vermutung**

$$r(k)=\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

k by k determinant

**r(2i+1) = 0  fuer  i = 1,2,3,4,…**

**r(k) sequence**

$$\left\langle \frac{1}{2}, \frac{1}{12}, 0, \frac{-1}{720}, 0, \frac{1}{30240}, 0, \frac{-1}{1209600}, 0, \frac{1}{47900160}, 0 \right.$$
$$\frac{-691}{1307674368000}, 0, \frac{1}{74724249600}, 0,$$
$$\frac{-3617}{1067062284288000}, 0, \frac{43867}{5109094217661709440000}, 0,$$
$$\left. \frac{-14611}{802857662698291200000}, 0, \ldots \right\rangle$$

$$r(2i) = \frac{B_{2i}}{(2i)!}$$

$$\lim_{i \to \infty} \frac{r(2i)}{r(2i+2)} = (2\pi)^2$$

---

**PT Vermutung**

$$s(k) = \begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 \\ 1 & 4 & 6 & 4 & 2 & 0 \\ 1 & 5 & 10 & 10 & 5 & 2 \\ 1 & 6 & 15 & 20 & 15 & 6 \end{vmatrix}$$

k by k determinant

**s(2i) = 0 fuer i = 1,2,3,4,…**

**s(k) sequence**

<1, 0, -2, 0, 16, 0, -272, 0, 7936, 0, -353792, 0, 22368256, 0, -1903757312, 0, 209865342976, 0, -29088885112832, 0,…>

**$s(2i-1) = 2^{2i}(2^{2i}-1)B_{2i}/2i$**

---

**$s(2i-1)/r(2i) = 2^{2i}(2^{2i}-1)(2i-1)!$**

---

# The Bernoulli wing

$$r(k) = \begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

k by k determinant

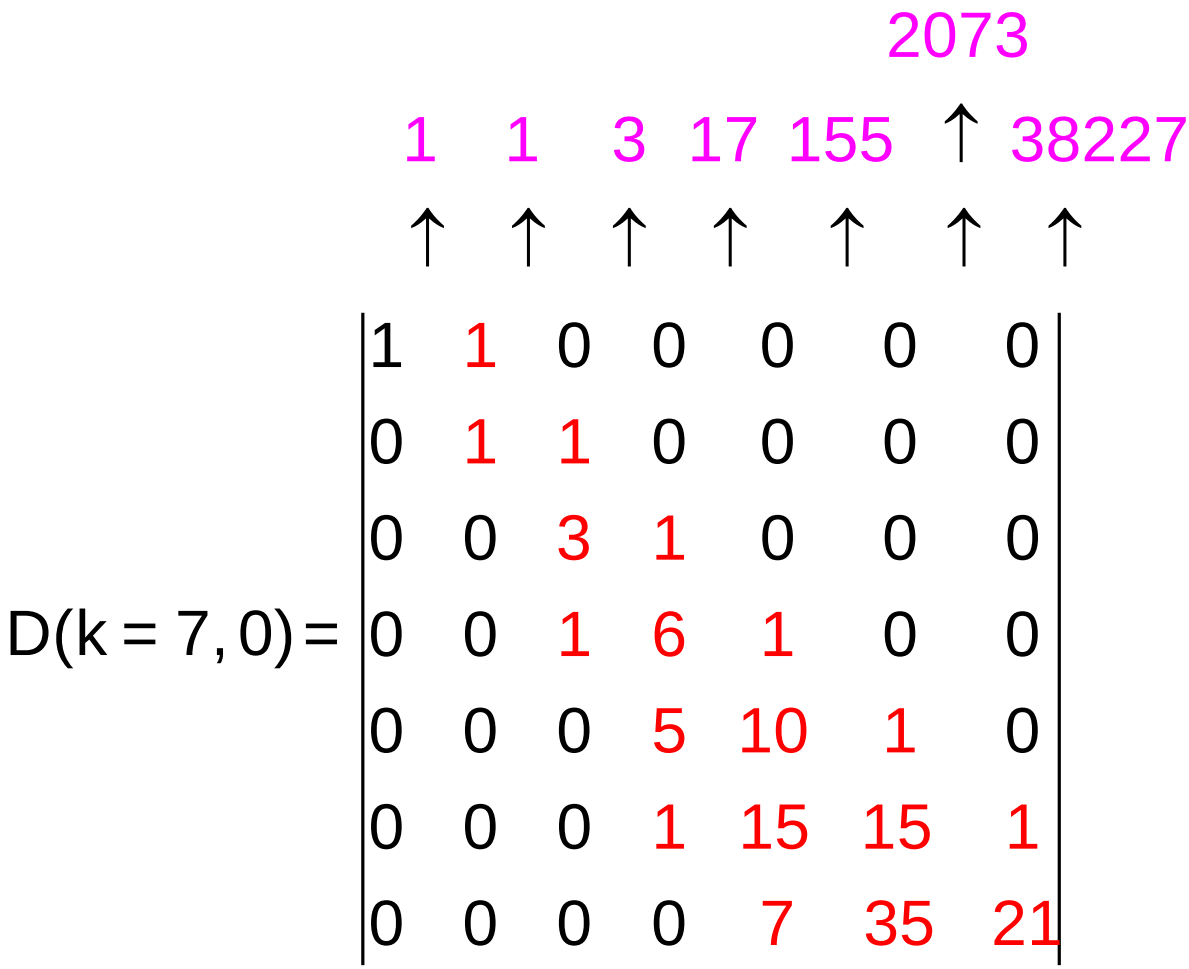
$$\Leftrightarrow$$

$$\frac{2^3-1}{2^4-1}\frac{1}{4!}\quad\frac{2^7-1}{2^8-1}\frac{17}{8!}\quad\frac{2^{11}-1}{2^{12}-1}\frac{2073}{12!}$$

$$\frac{2^1-1}{2^2-1}\frac{1}{2!}\uparrow\frac{2^5-1}{2^6-1}\frac{3}{6!}\uparrow\frac{2^9-1}{2^{10}-1}\frac{155}{10!}\uparrow\frac{2^{13}-1}{2^{14}-1}\frac{38227}{14!}$$

$$\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow\quad\uparrow$$

$$b(k=7)=\begin{vmatrix}
\frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & 0 \\
\frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 \\
\frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 \\
\frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 \\
\frac{1}{13!} & \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} \\
\frac{1}{15!} & \frac{1}{13!} & \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}
\end{vmatrix}$$




7 by 7 determinant

$$r(2i) = \frac{(-1)^{i-1}}{2^{2i}-2} b(i)$$

$$= \frac{(-1)^{i-1}}{2(2^{2i}-1)(2i)!} D(i,0)$$

$$s(2i-1) = (-1)^{i-1} \frac{2^{2i-2}}{i} D(i,0)$$

$$\begin{matrix} \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{matrix}$$

$$\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 & \rightarrow \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$e^{\pm x} = 1 \pm \begin{vmatrix} \frac{1}{1!} \end{vmatrix} x + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} x^2 \pm \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} x^4 \pm \ldots$$

$$= 1 \pm \frac{1}{1!}x + \frac{1}{2!}x^2 \pm \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \pm \ldots$$

$$= \begin{vmatrix} +x^0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ \mp x^1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ +x^2 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ \mp x^3 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ +x^4 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

$$= \begin{vmatrix} +x^0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ \mp x^1 & \frac{1}{0!} & 0 & 0 & 0 & \\ -x^2 & 0 & \frac{1}{0!} & 0 & 0 & \\ \mp x^3 & 0 & 0 & \frac{1}{0!} & 0 & \\ -x^4 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

$$\frac{1}{1!2!}\ \frac{1}{2!3!}\ \frac{1}{3!4!}\ \frac{1}{4!5!}\ \frac{1}{5!6!}\ \frac{1}{6!7!}$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

$$\begin{vmatrix}
\frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\
\frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & \\
\frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & \\
\frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & \\
\frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \\
\frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \\
\downarrow & & & & & & \searrow
\end{vmatrix}$$

Bessel functions of the first kind

$$\left.\begin{aligned} J_1(x) &= J_0'(x) \\ I_1(x) &= I_0'(x) \end{aligned}\right\} = \sum_{i=0}^{\infty} \frac{(\mp 1)^i}{i!(i+1)} \left(\frac{x}{2}\right)^{2i+1} =$$

$$\frac{x}{2} \mp \begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} \left(\frac{x}{2}\right)^3 + \begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} \left(\frac{x}{2}\right)^5 \mp \begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} \left(\frac{x}{2}\right)^7 + \ldots$$

http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html

$$\frac{J_1(x)}{J_0(x)} = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!(i+1)!}\left(\frac{x}{2}\right)^{2i+1}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!i!}\left(\frac{x}{2}\right)^{2i}} =$$

$$\begin{vmatrix} 0 & \frac{1}{0!1!} & \frac{1}{1!2!} & \frac{1}{2!3!} & \frac{1}{3!4!} & \frac{1}{4!5!} & \rightarrow \\ -\left(\frac{x}{2}\right)^1 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \frac{1}{4!4!} & \\ +\left(\frac{x}{2}\right)^3 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \\ -\left(\frac{x}{2}\right)^5 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \\ +\left(\frac{x}{2}\right)^7 & 0 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \\ -\left(\frac{x}{2}\right)^9 & 0 & 0 & 0 & 0 & \frac{1}{0!0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{I_1(x)}{I_0(x)} = \frac{\sum_{i=0}^{\infty} \frac{1}{i!(i+1)!}\left(\frac{x}{2}\right)^{2i+1}}{\sum_{i=0}^{\infty} \frac{1}{i!i!}\left(\frac{x}{2}\right)^{2i}} =$$

$$-\begin{vmatrix} 0 & \frac{1}{0!1!} & \frac{1}{1!2!} & \frac{1}{2!3!} & \frac{1}{3!4!} & \frac{1}{4!5!} & \rightarrow \\ \left(\frac{x}{2}\right)^{1} & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \frac{1}{4!4!} & \\ \left(\frac{x}{2}\right)^{3} & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \\ \left(\frac{x}{2}\right)^{5} & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \\ \left(\frac{x}{2}\right)^{7} & 0 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \\ \left(\frac{x}{2}\right)^{9} & 0 & 0 & 0 & 0 & \frac{1}{0!0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

**scf (einfache Kettenbruch) Darstellung fuer**

$$\frac{J_1(2)}{J_0(2)} = 2.57592032136822195685749678232....$$

$$= [2;1,1,2,1,3,1,4,1,5,1,6,1,7,1,8,1,9,1,10,$$
$$1,11,1,12,1,13,1,14,1,15,...]\text{?}$$

und

$$\frac{I_1(2)}{I_0(2)} = 0.69777465796400798200679059255 2...$$

$$= [0;1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,$$
$$17,18,19,20,21,22,23,24,25,26,27,28,...]$$

# Zum Vergleich

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!i!} x^{2i}} =$$

$$\begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\sum_{i=0}^{\infty} \frac{1}{(2i+1)!} x^{2i+1}}{\sum_{i=0}^{\infty} \frac{1}{(2i)!} x^{2i}} =$$

$$- \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

**scf (einfache Kettenbruch) Darstellung fuer**

**tan1** = 1.557407724654902230506974807 46…

= **[**1,1,1,3,1,5,1,7,1,9,1,11,1,13,1,15,1,17,1,

19,1,21,1,23,1, 25,1,27,1,29,…]

**tanh1** = 0.761594155955764888119458282605…

= **[**0, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23,
25, 27, 29, 31, 33, 35, 37, 39, 41, 43,
45,47, 49, 51, 53, 55, 57,…]

$$\frac{1}{J_0(x)} = \frac{1}{\sum_{i=0}^{\infty} \frac{(-1)^i}{i!i!}\left(\frac{x}{2}\right)^{2i}} =$$

$$\begin{vmatrix}
+\left(\frac{x}{2}\right)^{0} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \frac{1}{4!4!} & \frac{1}{5!5!} & \rightarrow \\
-\left(\frac{x}{2}\right)^{2} & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \frac{1}{4!4!} & \\
+\left(\frac{x}{2}\right)^{4} & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \frac{1}{3!3!} & \\
-\left(\frac{x}{2}\right)^{6} & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \frac{1}{2!2!} & \\
+\left(\frac{x}{2}\right)^{8} & 0 & 0 & 0 & \frac{1}{0!0!} & \frac{1}{1!1!} & \\
-\left(\frac{x}{2}\right)^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!0!} & \\
\downarrow & & & & & & \searrow
\end{vmatrix}$$

**Zum Vergleich**

$$\frac{1}{\cos x} = \frac{1}{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i}} =$$

$$\begin{vmatrix} +x^0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$\frac{J_1(2)}{J_0(2)} = 2.575920321368221956857496782 32....$$

$$= [2;1,1,2,1,3,1,4,1,5,1,6,1,7,1,8,1,9,1,10,$$
$$1,11,1,12,1,13,1,14,1,15,...]$$

und

$$\frac{I_1(2)}{I_0(2)} = 0.697774657964007982006790592552...$$

$$= [0;1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,$$
$$17,18,19,20,21,22,23,24,25,26,27,28,...]$$

**tan1** = 1.557407724654902230506974807 46…

**=[**1;1,1,3,1,5,1,7,1,9,1,11,1,13,1,15,1,17,1,
19,1,21,1,23,1, 25,1,27,1,29,…]

**tanh1** =0.761594155955764888119458282605…

**=[**0; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21,
23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43,
45,47, 49, 51, 53, 55, 57,…]

Und dazu

$$\frac{J_1(1)}{J_0(1)} = 0.57508091500430596049944339531 9...$$

$$= [0;1,1,2,1,4,1,6,1,8,1,10,1...]$$

und

$$\frac{I_1(1)}{I_0(1)} = 0.44638996589653450704768179519 3...$$

$$= [0;2,4,6,8,10,...]$$

**tan0.5** = 0.54630248984379051325517946578 0…

= **[**0;1,1,4,1,8,1,12,1,16,1,20,1,…]

**tanh0.5** = 0.46211715726000975850231848364…

= **[**0;2,6,10,14,18,22,26,30,…]

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