Unit 7
Under the roof of the 2by2 matrix chain product.
An attempt of a synopsis.

A multitude of computing, representation, and transformation problems, e.g.
- extended Euclidean algorithm,
- solution of the linear Diophantine equation,
- transformation of a continued fraction into a common fraction,
- representation of elementary functions by infinite 2by2 matrix chain products,
- Horner scheme,
- transformation of a nested expression into a common fraction,
- Euler transformation of a polynomial into a continued fraction,
- encoding and decoding in the 2by2 reduced Pythagorean triple tree,
- electrical two-port network analysis and synthesis
- decomposition of 2by2 coupling, chain, coding, … matrices into “atomic matrices”,

* can be transformed into a 2by2 matrix chain product representation m domain,
* can be so considered as structurally similar problems,
* and can be uniformly and simpler handled as in the original domain (compression/merging of neighboring matrix groups to a product-matrix, decomposition into factor-matrices, regrouping, transformation, tuning, window and parallel processing…).

Let me start with the **extended Euclidean algorithm**
* to find the gcd of two natural numbers a and b without prime factorization using the reduction formula

$$x = y*\mathrm{div}(x,y) + \mathrm{rest}(x,y) = y*\mathrm{int}\left(\frac{x}{y}\right) + x\mathrm{mod}y \; ; \; x,y \in \mathbb{N}$$

step by step from i.g. large numbers (a,b) to small numbers (gcd(a,b),0)

$u := a; \quad v := b; \quad \underline{w} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad i := 0$

$a_i := div(u,v) \rightarrow$

$\underline{w} := \begin{bmatrix} 0 & 1 \\ 1 & -div(u,v) \end{bmatrix} \underline{w}$

$\begin{bmatrix} u \\ v \end{bmatrix} := \begin{bmatrix} v \\ rest(u,v) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -div(u,v) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

$i := i + 1$

$v = 0$

$\gcd(a,b) := u$

$n := i - 1$

$scf\left(\frac{a}{b}\right) = [a_0;\ a_1,\ a_2, \cdots,\ a_n]$

$\underline{m} := \underline{w}$

$\begin{bmatrix} \gcd(a,b) \\ 0 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \underline{m} \begin{bmatrix} a \\ b \end{bmatrix}; \quad abs(\det \underline{m}) = 1$

$\frac{a}{\gcd(a,b)} := abs(m_{22}); \quad \frac{b}{\gcd(a,b)} := abs(m_{21})$

$\begin{bmatrix} x \\ y \end{bmatrix} = \underline{m}^T \begin{bmatrix} c \\ t \end{bmatrix}$ is solution of the linear Diophantine equation $\quad ax + by = c \cdot \gcd(a,b); \quad a,b \in \mathbb{N};$

$x, y, c, t \in \mathbb{Z}$

* to calculate simultaneously the expansion of the common

fraction a/b into a simple continued fraction scf and
* the 2by2 Euclidean transformation matrix $\underline{m}$,

$$\begin{pmatrix} \gcd(a,b) \\ 0 \end{pmatrix} = \underline{m} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

at the same time solution matrix of the linear Diophantine equation a*x + b*y = c*gcd(a,b)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{m}^{T} \begin{pmatrix} c \\ t \end{pmatrix} ; \; a,b \in \mathbb{N} \; ; \; c,x,y,t \in \mathbb{Z}$$

Example 1
a = 1272, b = 684
gcd(a,b) = 12
n = 3, scf(a/b) = [1;1,6,8]

$$\underline{m} = \begin{bmatrix} 7 & -1 \\ -57 & 106 \end{bmatrix} \overset{3}{=} \begin{bmatrix} 0 & 1 \\ 1 & -8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The 2by2 matrix and the 2by2 matrix chain product enter the stage, here together with the scf expansion.

$$\begin{bmatrix} 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 & -13 \\ -57 & 106 \end{bmatrix} \begin{bmatrix} 1272 \\ 684 \end{bmatrix}$$

$$\frac{a}{\gcd(a,b)} = 106 \; ; \quad \frac{b}{\gcd(a,b)} = 57$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & -57 \\ -13 & 106 \end{bmatrix} \begin{bmatrix} c \\ t \end{bmatrix} ; \; x = 7c - 57t \; ; \; y = -13c + 106t$$

is solution of the linear Diophantine equation
$1272x + 684y = 12c \; ; \quad x,y,c,t \in \mathbb{Z}$

e.g. decomposes the integer 12c into the two integer weighted summands 1272x and 684y controlled by the integer t.

For c=2 we have

t	x	y
-3	188	-344
-2	128	-238
-1	71	-132

0	14	-26
1	-43	80
2	-108	186
3	-157	292

Continued fraction to common fraction

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\ddots+\cfrac{b_n}{a_n}}}}=a_0+\frac{b_1\,|}{|\,a_1}+\frac{b_2\,|}{|\,a_2}+\frac{b_3\,|}{|\,a_3}+..+\frac{b_n\,|}{|\,a_n}\Rightarrow\frac{a}{b}$$

$$a_i,b_i,a,b\in \mathrm{N}$$

2 by 2 matrix chain product representation

$$\begin{bmatrix} & b \\ & a \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & a_0 \end{bmatrix}\begin{bmatrix} 0 & b_1 \\ 1 & a_1 \end{bmatrix}\begin{bmatrix} 0 & b_2 \\ 1 & a_2 \end{bmatrix}\begin{bmatrix} 0 & b_3 \\ 1 & a_3 \end{bmatrix}..\begin{bmatrix} 0 & b_n \\ 1 & a_n \end{bmatrix}$$

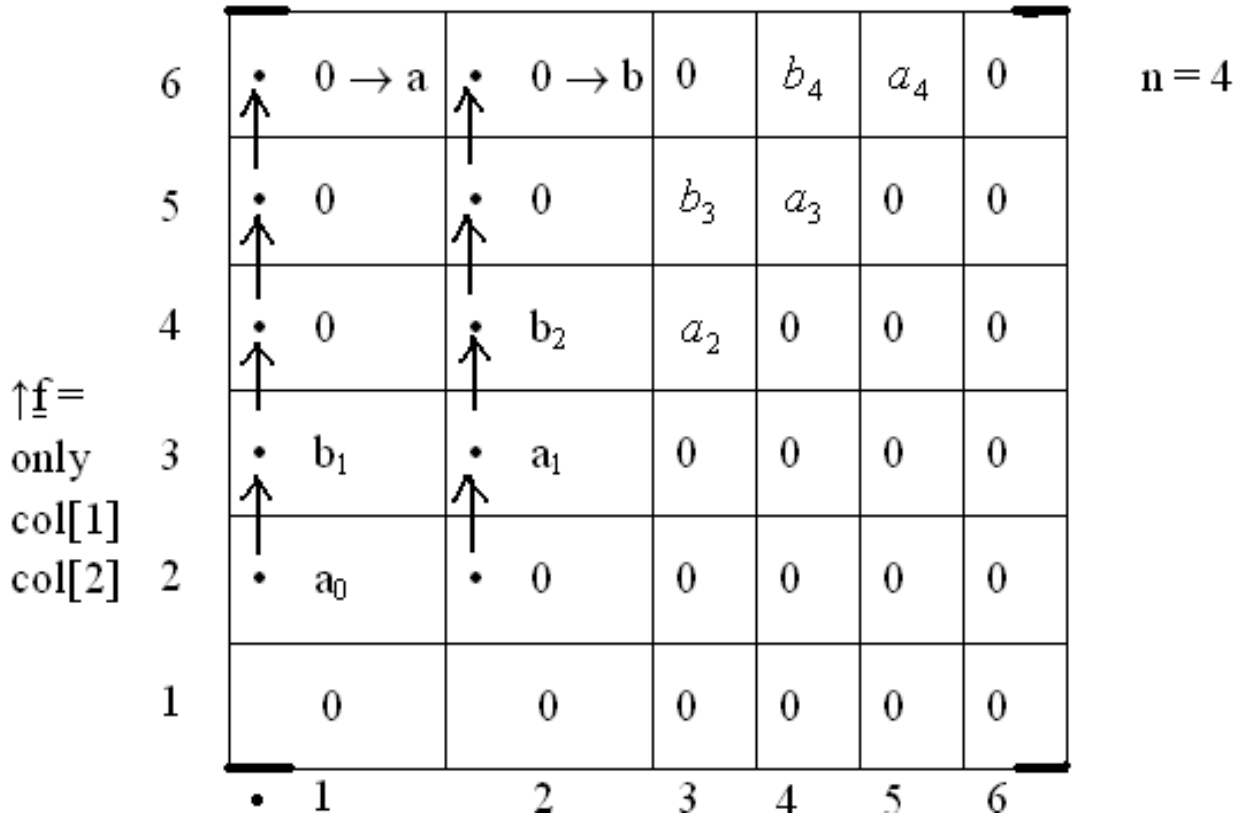
$$\begin{bmatrix} a & b \\ & \end{bmatrix}=\begin{bmatrix} a_n & b_n \\ 1 & 0 \end{bmatrix}..\begin{bmatrix} a_3 & b_3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}$$

Classical determinant notation:

$$\begin{vmatrix} a_0 & b_1 & 0 & 0 & 0 & 0 \\ -1 & a_1 & b_2 & 0 & 0 & 0 \\ 0 & -1 & a_2 & b_3 & 0 & 0 \\ 0 & 0 & -1 & a_3 & \ddots & 0 \\ 0 & 0 & 0 & -1 & \ddots & b_n \\ 0 & 0 & 0 & 0 & -1 & a_n \end{vmatrix}_{a\ b}=\begin{vmatrix} a_n & b_n & 0 & 0 & 0 & 0 \\ -1 & a_{n-1} & b_{n-1} & 0 & 0 & 0 \\ 0 & -1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & -1 & a_2 & b_2 & 0 \\ 0 & 0 & 0 & -1 & a_1 & b_1 \\ 0 & 0 & 0 & 0 & -1 & a_0 \end{vmatrix}^{b\ a} \qquad (14.1.2$$

Cartesian dynamic determinant notation over (R; +, ·, 0, 1),

only column [1] and [2] navigation



Example 2

$$1+\frac{1|}{|1}+\frac{1|}{|6}+\frac{1|}{|8}\Rightarrow\frac{a}{b}$$

$$\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}0&1\\1&6\end{bmatrix}\begin{bmatrix}0&1\\1&8\end{bmatrix}=\begin{bmatrix}7&b=5\\13&a=106\end{bmatrix}$$

$$\begin{bmatrix}8&1\\1&0\end{bmatrix}\begin{bmatrix}6&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}=\begin{bmatrix}a=106&b=57\\1\quad3&7\end{bmatrix}=(\underline{m}^{-1})^{T}$$

Example 3 e – experiment, **2 by 2 matrix tuning**

$$e = 2+\frac{1|}{|1}+\frac{1|}{|2}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|4}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|6}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|8}+\frac{1|}{|1}+..$$

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \end{array}\right]\underbrace{\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right]\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \end{array}\right]\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right]}_{\downarrow}\underbrace{\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right]\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 4 \end{array}\right]\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right]}_{\downarrow} \ldots$$

$$= \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \end{array}\right] \qquad \left[\begin{array}{c|c} 2 & 3 \\ \hline 3 & 4 \end{array}\right] \qquad \left[\begin{array}{c|c} 4 & 5 \\ \hline 5 & 6 \end{array}\right]$$

$$= \prod_{i=0}^{\infty}\left[\begin{array}{c|c} 2i & 2i+1 \\ \hline 2i+1 & 2i+2 \end{array}\right]$$

Example 4 How strange!

$$\frac{1|}{|2}+\frac{2|}{|3}+\frac{3|}{|4}+\frac{4|}{|5}+\frac{5|}{|6}+\frac{6|}{|7}+..=\frac{3-e}{e-2}$$

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right]\left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 2 \end{array}\right]\left[\begin{array}{c|c} 0 & 2 \\ \hline 1 & 3 \end{array}\right]\left[\begin{array}{c|c} 0 & 3 \\ \hline 1 & 4 \end{array}\right]\left[\begin{array}{c|c} 0 & 4 \\ \hline 1 & 5 \end{array}\right]..$$

$$\frac{2|}{|1}+\frac{3|}{|2}+\frac{4|}{|3}+\frac{5|}{|4}+\frac{6|}{|5}+\frac{7|}{|6}+..=1 \ !$$

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right]\left[\begin{array}{c|c} 0 & 2 \\ \hline 1 & 1 \end{array}\right]\left[\begin{array}{c|c} 0 & 3 \\ \hline 1 & 2 \end{array}\right]\left[\begin{array}{c|c} 0 & 4 \\ \hline 1 & 3 \end{array}\right]\left[\begin{array}{c|c} 0 & 5 \\ \hline 1 & 4 \end{array}\right]..$$

Example 5

$$e^k \Rightarrow \left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \prod_{i=0}^{\infty}\left[\begin{array}{c|c} 2i+1-k & 2i+1 \\ \hline 2i+1 & 2i+1+k \end{array}\right]$$

$$\sqrt[k]{e} \Rightarrow \left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \prod_{i=0}^{\infty}\left[\begin{array}{c|c} k(2i+1)-1 & k(2i+1) \\ \hline k(2i+1) & k(2i+1)+1 \end{array}\right]$$

Representation of elementary functions by infinite 2 by 2 matrix chain products

e^x & co.

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \prod_{i=1}^{\infty}\left[\begin{array}{c|c} 2i+1-x & 2i+1 \\ \hline 2i+1 & 2i+1+x \end{array}\right]$$

$$e^x = \frac{a}{b};\ e^{-x} = \frac{b}{a}$$

$$\sinh x = \frac{1}{2}\left(\frac{a}{b} - \frac{b}{a}\right);\ \cosh x = \frac{1}{2}\left(\frac{a}{b} + \frac{b}{a}\right)$$

$$\tanh\frac{x}{2} = \frac{a-b}{a+b};\ \tanh x = \frac{a^2-b^2}{a^2+b^2}$$

e^{2x} & co.

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \left[\begin{array}{c|c} 1 & 1-x \\ \hline 1 & 1+x \end{array}\right]\prod_{i=1}^{\infty}\left[\begin{array}{c|c} x^2 & (4i+1)x^2 \\ \hline 4i+1 & (4i-1)(4i+1)+x^2 \end{array}\right]$$

$$e^{2x} = \frac{a}{b};\ \tanh x = \frac{a-b}{a+b}$$

tanhx and tanx

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & x \end{array}\right]\prod_{i=1}^{\infty}\left[\begin{array}{c|c} \pm x^2 & \pm(4i+1)x^2 \\ \hline 4i-1 & (4i-1)(4i+1)\pm x^2 \end{array}\right]$$

$$\tanh x = \left(\frac{a}{b}\right)_+;\ \tan x = \left(\frac{a}{b}\right)_-$$

arctanx and artanhx

$$\left[\begin{array}{c|c} & b \\ \hline & a \end{array}\right] = \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & x \end{array}\right]\prod_{i=1}^{\infty}\left[\begin{array}{c|c} \pm(2i-1)^2x^2 & \pm(4i+1)(2i-1)^2x^2 \\ \hline 4i-1 & (4i-1)(4i+1)\pm(2i)^2x^2 \end{array}\right]$$

$$\arctan x = \left(\frac{a}{b}\right)_+;\quad x \geq 0$$

$$\operatorname{artanh} x = \left(\frac{a}{b}\right)_-;\quad 1 > x \geq 0$$

2 by 2 matrix chain product representation for the multiple

angle formulae of tan(x) and tanh(x)

$$\underline{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 0 & -(1 \pm a^2) \\ 1 & 2 \end{pmatrix}^{k-1} \quad ; \; a = \begin{matrix} \tan(x) \\ \tanh(x) \end{matrix} \text{ for } \begin{matrix} + \\ - \end{matrix}$$

$$\frac{m_{22}}{m_{12} + m_{22}} = \begin{matrix} \dfrac{\tan(kx)}{\tan(x)} \\ \dfrac{\tanh(kx)}{\tanh(x)} \end{matrix} \text{ for } \begin{matrix} + \\ - \end{matrix}$$

The same now for sin and cos!?

Horner's problem, Horner's algorithm

2 by 2 matrix chain product representation of a polynomial

$$p_n(x) = \sum_{i=0}^{n} a_i x^{n-i} \quad ; \quad x, a_i \in \mathbb{R}$$

via Horner's nested expression

$$p_n(x) = (..((((a_0)*x+a_1)*x+a_2)*x+a_3)*x+..+a_{n-1})*x+a_n$$

$$\overbrace{\underbrace{\underbrace{h_0}_{h_1 = h_0 * x + a_1}}_{h_2 = h_1 * x + a_2}}^{h_3 = h_2 * x + a_3}$$

$$\begin{bmatrix} 1 & p_n(x) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & a_1 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & a_2 \\ 0 & x \end{bmatrix} .. \begin{bmatrix} 1 & a_n \\ 0 & x \end{bmatrix}$$

$$\begin{bmatrix} 0 & p_n(x) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & a_n \\ 0 & 1 \end{bmatrix} .. \begin{bmatrix} x & a_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & a_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a_0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & p'_n(x) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & h_1 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & h_2 \\ 0 & x \end{bmatrix} .. \begin{bmatrix} 1 & h_{n-1} \\ 0 & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & h_i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & a_1 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & a_2 \\ 0 & x \end{bmatrix} .. \begin{bmatrix} 1 & a_i \\ 0 & x \end{bmatrix}$$

i=1,2,..,n-1

$$\begin{bmatrix} 1 & p''_n(x)/2! \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & g_1 \\ 0 & x \end{bmatrix}\begin{bmatrix} 1 & g_2 \\ 0 & x \end{bmatrix}..\begin{bmatrix} 1 & g_{n-1} \\ 0 & x \end{bmatrix}$$

$$\begin{bmatrix} 1 & g_i \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a_0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & h_1 \\ 0 & x \end{bmatrix}\begin{bmatrix} 1 & h_2 \\ 0 & x \end{bmatrix}..\begin{bmatrix} 1 & h_i \\ 0 & x \end{bmatrix}$$

i=1,2,..,n-2

How about gcd($p_n(x)$,p'(x))?
A glance sideways:
Cartesian dynamic determinant notation over (R; +, ·, 0, 1),
only column [1] navigation

$\underline{\uparrow f} =$ only col[1]

6	• $a_4 \to h_4 = p_4(x)$	0	0	0	x	0
5	• $a_3 \to h_3 \to g_4 = p'_4(x)$	0	0	x	0	0
4	• $a_2 \to h_2 \to g_3$	0	x	0	0	0
3	• $a_1 \to h_1 \to g_2$	x	0	0	0	0
2	• $a_0 \to h_0 \to g_1$	0	0	0	0	0
1	0	0	0	0	0	0
•	1	2	3	4	5	6

First run → $p_n(x)$, second run → $p'_n(x)$

Extension:
2 by 2 matrix chain product representation for transformation of a nested expression into a common fraction,

$$c_0 + \frac{a_1}{b_1}\left(c_1 + \frac{a_2}{b_2}\left(c_2 + \frac{a_3}{b_3}\left(c_3 + .. + \frac{a_n}{b_n}(c_n)..\right)\right)\right) \Rightarrow \frac{a}{b}$$

$a_i, b_i, c_i, a, b \in \mathrm{N}$

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & c_0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} 1 & c_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 1 & c_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 & 0 \\ 0 & b_3 \end{bmatrix} \begin{bmatrix} 1 & c_3 \\ 0 & 1 \end{bmatrix}}_{\downarrow} \cdot\cdot$$

$$= \begin{bmatrix} 1 & c_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_1 c_1 \\ 0 & b_1 \end{bmatrix} \quad \begin{bmatrix} a_2 & a_2 c_2 \\ 0 & b_2 \end{bmatrix} \quad \begin{bmatrix} a_3 & a_3 c_3 \\ 0 & b_3 \end{bmatrix}$$

$$\cdot\cdot \underbrace{\begin{bmatrix} a_n & 0 \\ 0 & b_n \end{bmatrix} \begin{bmatrix} 0 & c_n \\ 0 & 1 \end{bmatrix}}_{\downarrow}$$

$$\cdot\cdot \begin{bmatrix} 0 & a_n c_n \\ 0 & b_n \end{bmatrix}$$

Example 6 e – experiment, **2 by 2 matrix tuning!**

$$e = 1 + \frac{1}{1}(1 + \frac{1}{2}(1 + \frac{1}{3}(1 + \frac{1}{4}(1 + .. \rightarrow a_i = 1;\ b_i = i \qquad \downarrow$$

$$\begin{bmatrix} & a \\ & b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}}_{\downarrow} \cdot\cdot = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \prod_{i=1}^{\infty} \begin{bmatrix} 1 & 1 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \cdot 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 \\ 0 & 3 \cdot 4 \end{bmatrix} \cdot\cdot$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \prod_{i=1}^{\infty} \begin{bmatrix} 1 & 2i+1 \\ 0 & (2i-1)2i \end{bmatrix}$$

Heron in 2 by 2 matrix chain product outfit

$$\underline{m} := \begin{bmatrix} 0 & b \\ 1 & 2a \end{bmatrix};\quad a, b \in \mathbb{N}$$

$$\underline{m} := \underline{m}^2$$

$$m_{12} m_{22} \geq \frac{1}{\varepsilon}$$

$$\underline{m} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$sqr(\sqrt{a^2+b},\varepsilon) := a + \frac{m_{21}}{m_{22}}$$

Fast input follower

Input: b; x[0]; ε
x : = x[0]
y : = x
$\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} n-1 & n+1 \\ n+1 & n-1 \end{pmatrix} \begin{pmatrix} y^n \\ b^n \end{pmatrix}$
$x := y * \frac{u}{v}$
$\lvert x - y \rvert \leqslant \varepsilon$
? (b, ε) : = x
Output : (b,ε)

Experiment:
n = 2; b = 5; x[0] = 50

i	x[i]
0	50.0
1	17.10963455
2	6.966075316
3	5.044552888
4	5.000000871
5	5.000000001
6	5.0

After a short transient process (shape and duration depend on b, x[0], and ε) we obtain (b,ε) = b, i.e. output = input → dynamic input follower, strong attractor, cubic convergence.

Controllable decomposition of a positive real number b into two summands/factors

An application field of the dynamic input follower in the configuration carrier-rocket − shuttle is the decomposition (splitting) of the input b regarding a given expression containing k variables (simplest case: expression = v1 +v2) into k values of this variables (simplest case: va1 and va2) so that the value of the given expression is b for this k values of the variables (simplest case: va1 +va2 = b)
Experimental test:
b = 60,
expression = $v_1 + v_2$,
dynamic input follower recursion formula:

$$x[i+1] = x[i-1]\frac{1+3\left(\frac{b}{x[i-1]+x[i]}\right)^2}{3+\left(\frac{b}{x[i-1]+x[i]}\right)^2};\quad \text{i=1,2,3,...}$$

with the initial values
x[0] = x[1] =70

Record of the series $< x[i] >_{i=0}$

i	x[i]	x[i-1] + x[i]	b-comments
0	70.0		60.0
1	70.0	140.0	60.0
2	34.1025641	104.1025641	After a transient
3	41.94208978	76.04465388	process, b = 60 is
4	21.23010586	63.17219564	decomposited into
5	37.83559981	59.06570567	two summands,
6	21.90704831	59.74264812	21.86.. and 38.14..,
7	38.16226875	60.06931706	its sum is 60.
8	21.85651817	60.01878692	Decomposition
9	38.13838161	59.99489978	depends from b,
10	21.86023441	59.99861602	x[0], x[1]; can be
11	38.14014112	60.00037553	changed by
	→ +4 =		← injection
	42.14014112	64.00037553	in x[i]
12	20.49522576	62.63536688	
13	40.36790677	60.86313253	
14	20.20458676	60.57249353	
15	39.98638298	60.19096974	
16	20.1404832	60.12686618	
17	39.90201282	60.04249602	60.0
18	20.12622846	60.02824128	60.0
19	39.88324026	60.00946872	65.0 ← change of b
20	21.79720008	61.68044034	65.0
21	42.02819095	63.82539103	New transient process
22	22.19831065	64.2265016	
23	42.5343304	64.73264105	65.0
24	22.2899938	64.8243242	Decomposition of b into

25	42.64959946	64.93959326	summands 22.31.. and 42.68..
26	22.31072793	64.96032739	
27	42.67564647	64.9863744	
28	22.3154058	64.99105227	
29	42.68152188	64.99692768	
30	22.31646063	64.99798251	
31	42.68284647	64.9993073	65.0
		x[i-1]*x[i]	← Change of expression
		952.5300673	65.0
32	7.531049432	321.4466282	Fast factorization of b, cubic convergence!
33	15.75810547	118.6750712	
34	4.336007006	68.32725572	
35	14.99121665	65.00202042	
36	4.335872234	65.00000003	

How about integer factorization of $b \in \mathbb{N}$, esp. semiprime (= product of two primes)? → security guarantee of the RSA public key based cryptosystems.

Pythagorean triples, Pythagorean triple ternary tree structure, Pythagorean triple ternary coding algorithm, and beyond

The set $\prod$ of primitive Pythagorean triples

(a,b,c); $a, b, c \in \mathbb{N}$

$a < b < c$

$\gcd(a,b,c) = 1$

$a^2 + b^2 = c^2$

f. e. (3, 4, 5)
(5, 12, 13)
(8, 15, 17)
(20, 21, 29)

is countably infinite.

$\prod$ can be structured hierarchically by a complete ternary tree, e.g. using the following one-to-one mapping:

$$(a,b,c) \leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix};\ u,v \in \mathbb{N} \text{ and odd}$$

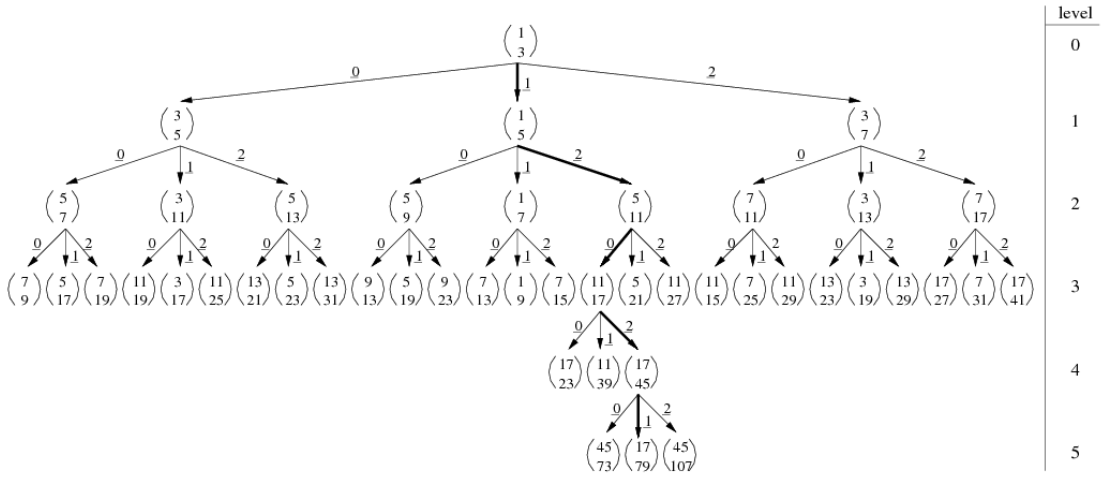
$u < v$

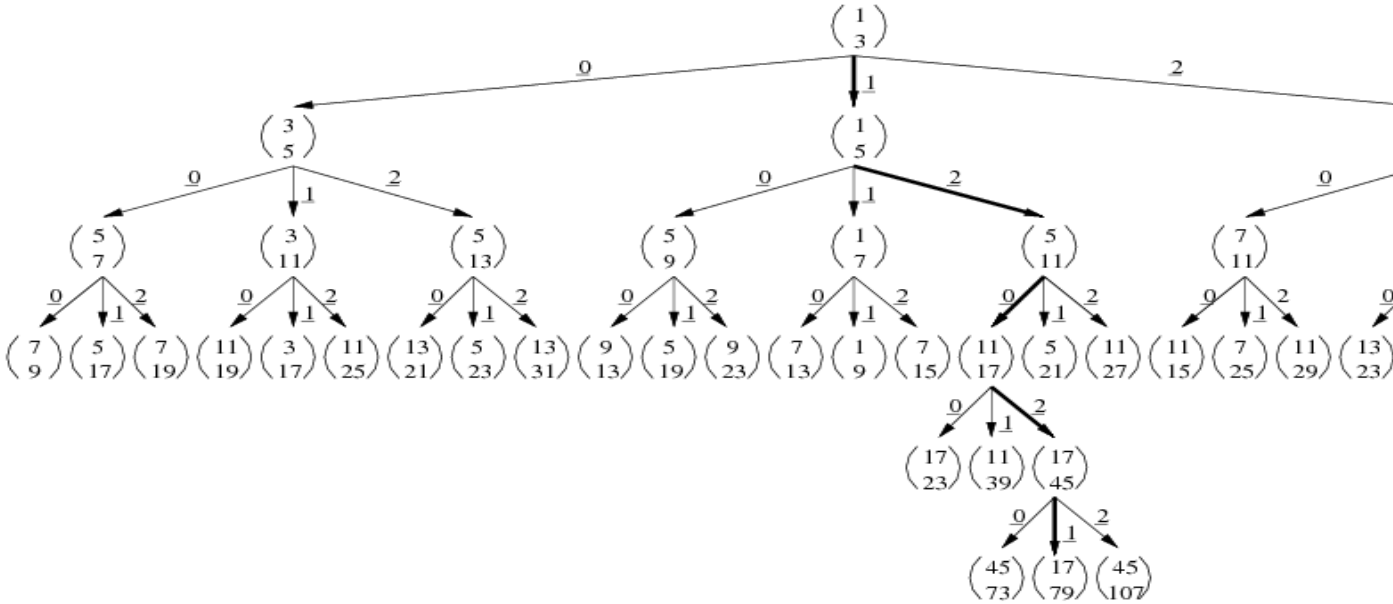
$$\gcd(u,v) = 1$$

with

the mapping equations	examples			
$\left.\begin{matrix} u \\ v \end{matrix}\right\} \sqrt{c \mp select \quad even \quad (a,b)}$	1 3	1 5	3 5	3 7
$a = \frac{1}{2}\min(v^2 - u^2, 2uv)$	3	5	8	20
$b = \frac{1}{2}\max(v^2 - u^2, 2uv)$	4	12	15	21
$c = \frac{1}{2}(v^2 + u^2)$	5	13	17	29

Each vector $\begin{pmatrix} u \\ v \end{pmatrix}$ is then located unique in a complete labeled Ternary three as shown detailed in fig. 1





$$\underline{0} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \underline{1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \underline{2} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Encoding: $\binom{u}{v} \to \underline{w} = \underline{m}(l) \ldots \underline{m}(3)\ \underline{m}(2)\ \underline{m}(1)$

$\uparrow$

start (bottom up)

Decoding: $\underline{w} \to \binom{u}{v} = \underline{w}\binom{1}{3}$

Fig. 1: $\Pi\binom{u}{v}$ ternary tree

The tree fig.1 is recursively structured.

$$\underline{0} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \underline{1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \underline{2} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

Encoding: $\binom{u}{v} \rightarrow \underline{w} = \underline{m}(l) \ldots \underline{m}(3)\ \underline{m}(2)\ \underline{m}(1)$

$\uparrow$

start (bottom up)

Decoding: $\underline{w} \rightarrow \binom{u}{v} = \underline{w} \binom{1}{3}$

Fig. 1: $\Pi \binom{u}{v}$ ternary tree

The tree fig.1 is recursively structured.

Beginning with the source vector (vertex) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (level 0), each vector (vertex) $\begin{pmatrix} u \\ v \end{pmatrix}$ on each level i has three successors on the next level i + 1, as described in fig. 2.

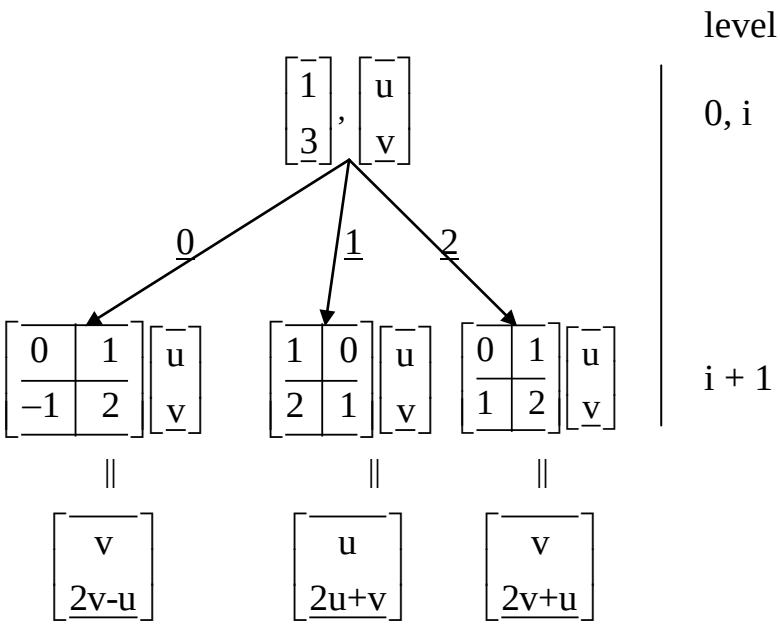


Fig. 2 : $\prod \begin{pmatrix} u \\ v \end{pmatrix}$ ternary tree grammar (starting symbol $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, substitution rule)

Using the tree grammar fig.2, each vector $\begin{pmatrix} u \\ v \end{pmatrix}$ (and with that each Pythagorean triple (a, b, c) via equ.(2)) can be calculated from the source vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (source triple (3, 4, 5)) and the accompanying three 2 by 2 coding matrices chain product.

$$\left.\begin{aligned}
\underline{0} &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \\
\underline{0}^{-1} &= \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
\underline{1} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
\underline{1}^{-1} &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\
\underline{2} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
\underline{2}^{-1} &= \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
\end{aligned}\right\} \text{decomposition of the 2 by 2 coding matrices into "elementary particles"} \qquad (23)$$

With other words: the position of each vector $\begin{pmatrix} u \\ v \end{pmatrix}$ in the tree can be described (encoded) uniquely by a ternary codeword

$\underline{w} = \underline{m}(l)..\underline{m}(3)\underline{m}(2)\underline{m}(1)$; $\underline{m(i)} \in \{\underline{0},\underline{1},\underline{2}\}$

of the length l (level of $\begin{pmatrix} u \\ v \end{pmatrix}$ in the tree). $\begin{pmatrix} u \\ v \end{pmatrix}$ is then given by the corresponding matrix chain product

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underline{w}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \underline{m}(l)..\underline{m}(3)\underline{m}(2)\underline{m}(1)\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

(decoding).

The decoding problem:
- given $\underline{w} = \underline{m}(l)..\underline{m}(3)\underline{m}(2)\underline{m}(1)$,
- find $\begin{pmatrix} u \\ v \end{pmatrix}$

is simple to solve:
- via matrix multiplication,
- go from source vector (vertex) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ along $\underline{m}(1)$, $\underline{m}(2)$, $\underline{m}(3)$,...,

  $\underline{m}(l)$ to vector (vertex) $\begin{pmatrix} u \\ v \end{pmatrix}$ (arrowed path in fig.1).

But how about <u>encoding:</u> $\begin{pmatrix} u \\ v \end{pmatrix}$

given
- $\begin{pmatrix} u \\ v \end{pmatrix}$,
- find $\underline{w} = \underline{m}(l)..\underline{m}(3)\underline{m}(2)\underline{m}(1)$,

that means
- the position of $\begin{pmatrix} u \\ v \end{pmatrix}$ in the coding tree fig.1
- the way from $\begin{pmatrix} u \\ v \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ step by step (level by level) through the coding tree fig.1.

This is an algorithmic challenge!
To solve this problem, we need an algorithmic idea to solve one step: from level i + 1 back to the level i in the right, unique, and simplest way!

That means: find a local operator that delivers for a given $\begin{pmatrix} u \\ v \end{pmatrix}$ the ancestor $\begin{pmatrix} u^* \\ v^* \end{pmatrix}$ in the coding tree and so its accompanying edge weight

$\underline{z} \in \{\underline{0},\underline{1},\underline{2}\}$ from $\begin{pmatrix} u* \\ v* \end{pmatrix}$ to $\begin{pmatrix} u \\ v \end{pmatrix}$ (see fig.2):

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underline{z} \begin{pmatrix} u* \\ v* \end{pmatrix} \tag{26}$$

Now the pool position!
For

$$\underline{1}^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{27}$$

we have with equ.(23)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{1}^{-1} \underline{z} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{cases} \underline{1}^{-1}\underline{0} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{pmatrix} v* \\ -u* \end{pmatrix} & \underline{z} = \underline{0} \\ \underline{1}^{-1}\underline{1} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{pmatrix} u* \\ v* \end{pmatrix} & \textit{for} \quad \underline{z} = \underline{1} \\ \underline{1}^{-1}\underline{2} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u* \\ v* \end{pmatrix} = \begin{pmatrix} v* \\ u* \end{pmatrix} & \underline{z} = \underline{2} \end{cases} \tag{28}$$

the three cases (way $\underline{0}$ or $\underline{1}$ or $\underline{2}$) are so distinguishable uniquely and very simple:

$y < 0 \quad \leftrightarrow \quad \underline{z} = \underline{0}$

$y > 0$ and $x < y \quad \leftrightarrow \quad \underline{z} = \underline{1}$

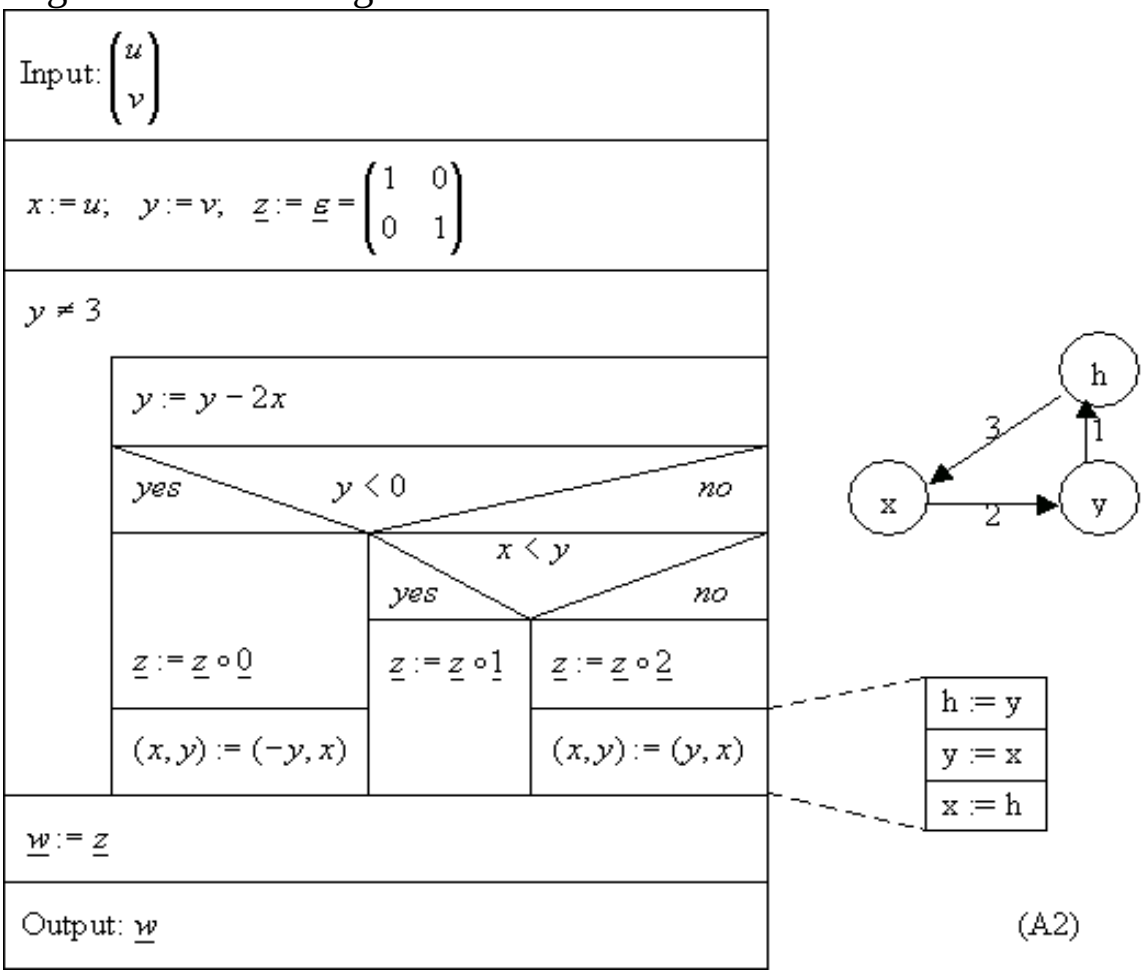
$y > 0$ and $x > y \quad \leftrightarrow \quad \underline{z} = \underline{2}$ (29)

Now we use this local operator starting with the given $\begin{pmatrix} u \\ v \end{pmatrix}$ in a loop exhaustively up to reaching the source $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, this navigation gives $\underline{w}$ in the sequence $\underline{m}(l),..,\underline{m}(3), \underline{m}(2), \underline{m}(1)$ (bottom-up encoding).

Our local operator $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{1}^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$ is here local calculator (calculation of

the ancestor $\begin{pmatrix} u* \\ v* \end{pmatrix}$ of $\begin{pmatrix} u \\ v \end{pmatrix}$) and at the same time also local data controlled navigator (go from $\begin{pmatrix} u \\ v \end{pmatrix}$ to the ancestor $\begin{pmatrix} u* \\ v* \end{pmatrix}$ in the coding tree via the edge with the weight $\underline{z}$ (a kind of greedy strategy).

Algorithm in structogram notation:



Example: $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 17 \\ 23 \end{pmatrix}$

i	x	y	$\underline{z}$
0	17	23	$\underline{\varepsilon}$
1	17 11	-11 17	$\underline{\varepsilon}\ \underline{0} = \underline{0}$
2	11 5	-5 11	$\underline{0}\ \underline{0}$
3	5	1	$\underline{0}\ \underline{0}\ \underline{2}$

	1	5	
4	1 1	3 3 stop	$\underline{0}\,\underline{0}\,\underline{2}\,\underline{1} = \underline{w}$ $\underline{0}\,\underline{0}\,\underline{2}\,\underline{1}\binom{1}{3} = \underline{\binom{17}{23}}$

If we continue the $\prod\binom{u}{v}$ ternary tree fig.1 for $i \to \infty$, we get from level to level increasingly larger values for u and v, but the values of the quotient $\frac{u}{v}$ remain in the interval $\left[\frac{1}{2i+3}, \frac{2i+1}{2i+3}\right]$. Through them, the values of $\frac{u}{v}$ for the left-justified strings j = 0, 1, 2, 3, .. of the ternary tree fig. 3 approaches limits as i approaches infinity.

The left sided series of these limits is identical with the series of the values of $\frac{u}{v}$ for the flesh left linear string representation of the $\frac{u}{v}$ tree level by level with level doubling (see fig. 3 and fig.1).

With other words: The fig. 3 left-justified $\frac{u}{v}$ infinity margin of the ternary tree is identical with its $\frac{u}{v}$ flesh left level wise doubled linear string representation!

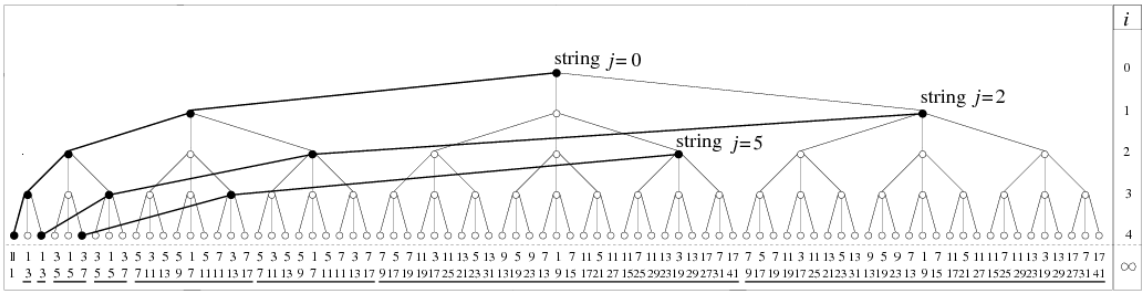


Fig. 2: $\binom{u}{v}$ ternary tree with left–justified strings $j = 0$, $j = 2$, and $j = 5$

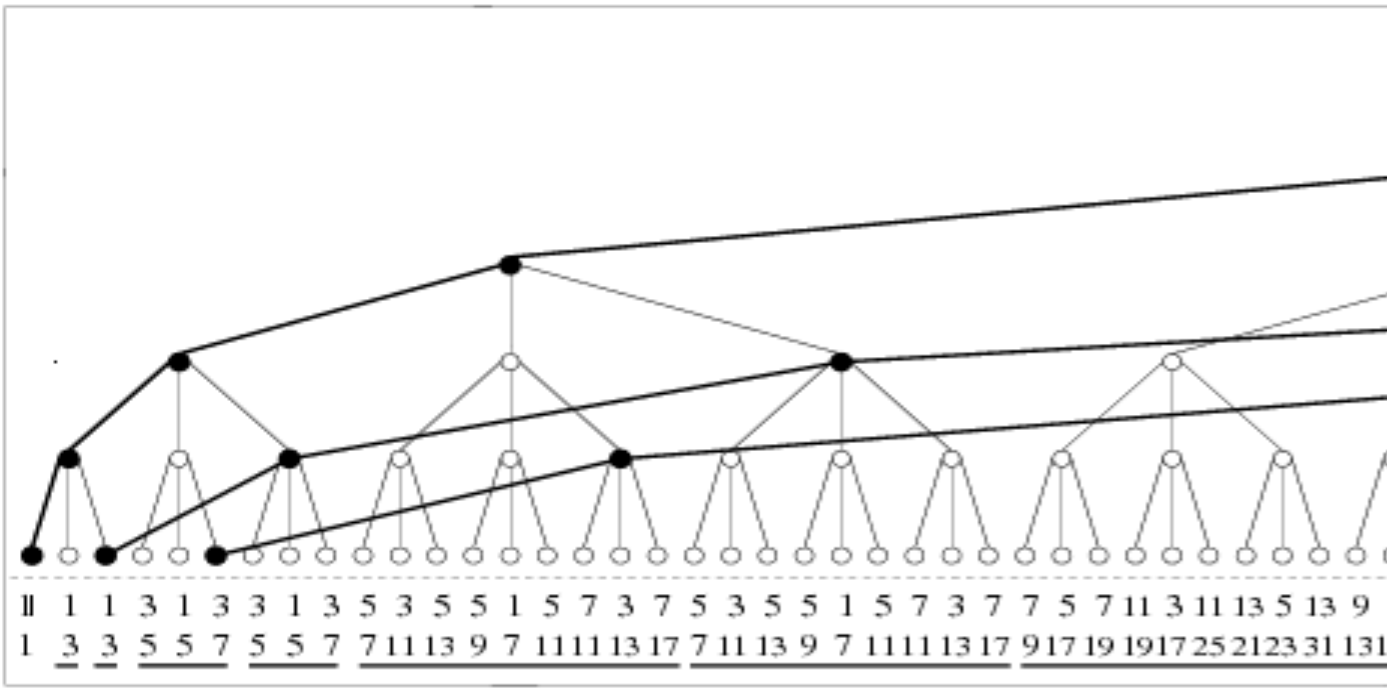


Fig. 2: $\binom{u}{v}$ ternary tree with left–justified strings $j = 0,\ j =$

The matrix chain product encoding of $\begin{pmatrix} u \\ v \end{pmatrix}$ given by

$$\left(\frac{u}{v}\right) = \underline{w}\left(\frac{1}{3}\right) \rightarrow \begin{pmatrix} \square & u \\ \square & v \end{pmatrix} = \underline{w}\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

can be transformed into the matrix chain product notation of the representation of $\begin{pmatrix} u \\ v \end{pmatrix}$ by a simple continued fraction

$$\frac{v}{u} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} = [a_0; a_1, a_2, ..., a_n]\,;\ (\forall i)\, a_i \in \mathbb{N}$$

using the following elementary decomposition and merging rules for 2 by 2 matrix product

- $$\begin{pmatrix} 1 & 0 \\ a_i & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (7)$$
- $$\begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_i + a_j \end{pmatrix} \quad (8)$$
- $$\begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_i - 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j - 1 \end{pmatrix} \quad (9)$$

in a suitable way.

Example 1

$\binom{u}{v} = \binom{17}{79}$

$$\begin{pmatrix} \square & 17 \\ \square & 79 \end{pmatrix} = \underline{1}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{1} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{(7)} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}}_{(9)} \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{(7)} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \overbrace{\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{(8)} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}} \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{(8)} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}}$$

$$= \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 5 \end{pmatrix}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$[a_1 = 4,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ a_5 = 5] = \frac{17}{79}$$

Example 2

$\binom{u}{v} = \binom{13}{75}$

$$\begin{pmatrix} \square & 13 \\ \square & 75 \end{pmatrix} = \underline{1}\,\underline{1}\,\underline{0}\,\underline{1}\,\underline{2} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{(7)} \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{(7)} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{(7)} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \overbrace{\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{(8)} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}}_{(8)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}}_{(9)} \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{(8)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}}_{(8)} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$= \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 5 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$[a_1 = 5,\ a_2 = 1,\ a_3 = 3,\ a_4 = 3] = \frac{13}{75}$$

☐ 1 Find a simple transformation algorithm

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underline{w} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} \square & u \\ \square & v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_3 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

$$\begin{pmatrix} 17 \\ 79 \end{pmatrix} = \underline{1}\underline{2}\underline{0}\underline{2}\underline{1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 37 & 14 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \Leftrightarrow \underline{w} \in \{\underline{0}, \underline{1}, \underline{2}\}^*$$