Unit 6 Series and triangular determinants An overview

6.1 Pascal triangular determinants and the sums of powers of integers

$$\sum_{i=1}^{n} i^{m=6} = \frac{\begin{vmatrix} (-n)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^3 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^4 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^5 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^6 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^7 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$\sum_{i=1}^{n}(-1)^{n-i}i^{m=6} = \frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ n^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ n^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ n^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ n^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ n^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^1 & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^2 & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^3 & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^4 & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^5 & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^6 & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 3 & 0 & 0 & 0 \\ -1 & 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 1 & 5 & 10 & 10 & 5 & 0 \\ -1 & 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = (n=7)!$$

(n=7) by (n=7) determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = \frac{(n=7)!}{2};\ n \geq 2$$

(n=7) by (n=7) determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 0 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ 0 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = (2k+1=7)!* B_{2k=6}$$

$(2k+1=7)$ by $(2k+1=7)$ determinant

$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = (2k+1=7)!* B_{2k=6}$$

$(2k=6)$ by $(2k=6)$ determinant

B_{2k} – Bernoulli numbers, see

http://mathworld.wolfram.com/BernoulliNumber.html

Interpretieren Sie den Grenzwert Λ des nachstehenden Determinantenquotienten $\Lambda(k)$

$$\Lambda(k) = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ -1 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 0 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ 0 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

$(2k+1=7)$ by $(2k+1=7)$ determinants

$$\Lambda = \lim_{k \to \infty} \Lambda(k) = 42.612923374243529927...$$

6.2 Triangular determinants and powers of infinite power series

Given $\sum_{i=0}^{\infty} a_i x^i$.

Find $\left(\sum_{i=0}^{\infty} a_i x^i \right)^n$; $n = -1, \pm 2, \pm \frac{1}{2}$

in determinant representation.

Solutions

$$\frac{\sum_{i=0}^{\infty} b_i x^i}{\sum_{i=0}^{\infty} a_i x^i} = -\begin{vmatrix} 0 & b_0 & b_1 & b_2 & b_3 & \rightarrow \\ x^0 & 1 & a_1 & a_2 & a_3 & \\ x^1 & 0 & 1 & a_1 & a_2 & \\ x^2 & 0 & 0 & 1 & a_1 & \\ x^3 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix} ; a_0 = 1$$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^n =$$

$$\begin{vmatrix}
x^0 & f(1,0)\,a_1 & f(2,0)\,a_2 & f(3,0)\,a_3 & f(4,0)\,a_4 & f(5,0)\,a_5 & f(6,0)\,a_6 & \rightarrow \\
x^1 & 1 & f(1,1)\,a_1 & f(2,1)\,a_2 & f(3,1)\,a_3 & f(4,1)\,a_4 & f(5,1)\,a_5 & \\
x^2 & 0 & 1 & f(1,2)\,a_1 & f(2,2)\,a_2 & f(3,2)\,a_3 & f(4,2)\,a_4 & \\
x^3 & 0 & 0 & 1 & f(1,3)\,a_1 & f(2,3)\,a_2 & f(3,3)\,a_3 & \\
x^4 & 0 & 0 & 0 & 1 & f(1,4)\,a_1 & f(2,4)\,a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & f(1,5)\,a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

$$a_0 = 1 \; ; \; n \in \mathbb{Q} \; ; \; f(i,j) = -\frac{in - j}{i + j} \; ; \; f(i,j)_{n=-1} = 1$$

Special cases

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^1 =$$

$$\begin{vmatrix}
x^0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & \rightarrow \\
x^1 & 1 & 0 & -\frac{1}{3}a_2 & -\frac{1}{2}a_3 & -\frac{3}{5}a_4 & -\frac{2}{3}a_5 & \\
x^2 & 0 & 1 & \frac{1}{3}a_1 & 0 & -\frac{1}{5}a_3 & -\frac{1}{3}a_4 & \\
x^3 & 0 & 0 & 1 & \frac{1}{2}a_1 & \frac{1}{5}a_2 & 0 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{1}{5}a_1 & \frac{1}{3}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} =$$

$$\begin{vmatrix}
x^0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & \rightarrow \\
x^1 & 1 & 0 & 0 & 0 & 0 & 0 & \\
x^2 & 0 & 1 & 0 & 0 & 0 & 0 & \\
x^3 & 0 & 0 & 1 & 0 & 0 & 0 & \\
x^4 & 0 & 0 & 0 & 1 & 0 & 0 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & 0 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

$a_0 = 1$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^{-1} =$$

$$\begin{vmatrix}
x^0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & \rightarrow \\
x^1 & 1 & a_1 & a_2 & a_3 & a_4 & a_5 & \\
x^2 & 0 & 1 & a_1 & a_2 & a_3 & a_4 & \\
x^3 & 0 & 0 & 1 & a_1 & a_2 & a_3 & \\
x^4 & 0 & 0 & 0 & 1 & a_1 & a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}$$

$a_0 = 1$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^{2} =$$

$$\begin{vmatrix}
x^0 & -2a_1 & -2a_2 & -2a_3 & -2a_4 & -2a_5 & -2a_6 & \rightarrow \\
x^1 & 1 & -\frac{1}{2}a_1 & -\frac{3}{3}a_2 & -\frac{5}{4}a_3 & -\frac{7}{5}a_4 & -\frac{9}{6}a_5 & \\
x^2 & 0 & 1 & -\frac{0}{3}a_1 & -\frac{2}{4}a_2 & -\frac{4}{5}a_3 & -\frac{6}{6}a_4 & \\
x^3 & 0 & 0 & 1 & -\frac{-1}{4}a_1 & -\frac{1}{5}a_2 & -\frac{3}{6}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{-2}{5}a_1 & -\frac{0}{6}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{-3}{6}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} =$$

$$
\begin{vmatrix}
x^0 & -2a_1 & -2a_2 & -2a_3 & -2a_4 & -2a_5 & -2a_6 & \rightarrow \\
x^1 & 1 & -\frac{1}{2}a_1 & -a_2 & -\frac{5}{4}a_3 & -\frac{7}{5}a_4 & -\frac{3}{2}a_5 & \\
x^2 & 0 & 1 & 0 & -\frac{1}{2}a_2 & -\frac{4}{5}a_3 & -a_4 & \\
x^3 & 0 & 0 & 1 & \frac{1}{4}a_1 & -\frac{1}{5}a_2 & -\frac{1}{2}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{2}{5}a_1 & 0 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix}
$$

$a_0 = 1$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^{\frac{1}{2}} =$$

$$\begin{vmatrix}
x^0 & -\frac{1}{2}a_1 & -\frac{1}{2}a_2 & -\frac{1}{2}a_3 & -\frac{1}{2}a_4 & -\frac{1}{2}a_5 & -\frac{1}{2}a_6 & \rightarrow \\
x^1 & 1 & -\frac{-1}{4}a_1 & -\frac{0}{6}a_2 & -\frac{1}{8}a_3 & -\frac{2}{10}a_4 & -\frac{3}{12}a_5 & \\
x^2 & 0 & 1 & -\frac{-3}{6}a_1 & -\frac{-2}{8}a_2 & -\frac{-1}{10}a_3 & -\frac{0}{12}a_4 & \\
x^3 & 0 & 0 & 1 & -\frac{-5}{8}a_1 & -\frac{-4}{10}a_2 & -\frac{-3}{12}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & -\frac{-7}{10}a_1 & -\frac{-6}{12}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & -\frac{-9}{12}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} =$$

$$
\left|\begin{array}{cccccc|c}
x^0 & -\frac{1}{2}a_1 & -\frac{1}{2}a_2 & -\frac{1}{2}a_3 & -\frac{1}{2}a_4 & -\frac{1}{2}a_5 & -\frac{1}{2}a_6 & \rightarrow \\
x^1 & 1 & \frac{1}{4}a_1 & 0 & -\frac{1}{8}a_3 & -\frac{1}{5}a_4 & -\frac{1}{4}a_5 & \\
x^2 & 0 & 1 & \frac{1}{2}a_1 & \frac{1}{4}a_2 & \frac{1}{10}a_3 & 0 & \\
x^3 & 0 & 0 & 1 & \frac{5}{8}a_1 & \frac{2}{5}a_2 & \frac{1}{4}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{7}{10}a_1 & \frac{1}{2}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{3}{4}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{array}\right|
$$

$a_0 = 1$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^{-\frac{1}{2}} =$$

$$\begin{vmatrix}
x^0 & \frac{1}{2}a_1 & \frac{1}{2}a_2 & \frac{1}{2}a_3 & \frac{1}{2}a_4 & \frac{1}{2}a_5 & \frac{1}{2}a_6 & \rightarrow \\
x^1 & 1 & \frac{3}{4}a_1 & \frac{4}{6}a_2 & \frac{5}{8}a_3 & \frac{6}{10}a_4 & \frac{7}{12}a_5 & \\
x^2 & 0 & 1 & \frac{5}{6}a_1 & \frac{6}{8}a_2 & \frac{7}{10}a_3 & \frac{8}{12}a_4 & \\
x^3 & 0 & 0 & 1 & \frac{7}{8}a_1 & \frac{8}{10}a_2 & \frac{9}{12}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{9}{10}a_1 & \frac{10}{12}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{11}{12}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} =$$

$$
\left|\begin{array}{cccccc}
x^0 & \frac{1}{2}a_1 & \frac{1}{2}a_2 & \frac{1}{2}a_3 & \frac{1}{2}a_4 & \frac{1}{2}a_5 & \frac{1}{2}a_6 & \rightarrow \\
x^1 & 1 & \frac{3}{4}a_1 & \frac{2}{3}a_2 & \frac{5}{8}a_3 & \frac{3}{5}a_4 & \frac{7}{12}a_5 & \\
x^2 & 0 & 1 & \frac{5}{6}a_1 & \frac{3}{4}a_2 & \frac{7}{10}a_3 & \frac{2}{3}a_4 & \\
x^3 & 0 & 0 & 1 & \frac{7}{8}a_1 & \frac{4}{5}a_2 & \frac{1}{3}a_3 & \\
x^4 & 0 & 0 & 0 & 1 & \frac{9}{10}a_1 & \frac{5}{6}a_2 & \\
x^5 & 0 & 0 & 0 & 0 & 1 & \frac{11}{12}a_1 & \\
x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\
\downarrow & & & & & & & \searrow
\end{array}\right|
$$

$a_0 = 1$

$$
e^{\sum_{i=1}^{\infty} a_i x^i} =
$$

$$\begin{vmatrix} x^0 & f(1,0)\,a_1 & f(2,0)\,a_2 & f(3,0)\,a_3 & f(4,0)\,a_4 & f(5,0)\,a_5 & f(6,0)\,a_6 & \rightarrow \\ x^1 & 1 & f(1,1)\,a_1 & f(2,1)\,a_2 & f(3,1)\,a_3 & f(4,1)\,a_4 & f(5,1)\,a_5 & \\ x^2 & 0 & 1 & f(1,2)\,a_1 & f(2,2)\,a_2 & f(3,2)\,a_3 & f(4,2)\,a_4 & \\ x^3 & 0 & 0 & 1 & f(1,3)\,a_1 & f(2,3)\,a_2 & f(3,3)\,a_3 & \\ x^4 & 0 & 0 & 0 & 1 & f(1,4)\,a_1 & f(2,4)\,a_2 & \\ x^5 & 0 & 0 & 0 & 0 & 1 & f(1,5)\,a_1 & \\ x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & & \searrow \end{vmatrix}$$

$$f(i,j) = -\frac{i}{i+j}$$

Special cases

** e^x $$f(i,j)a_i = \begin{cases} -\dfrac{1}{1+j} & \text{for } \begin{cases} i = 1 \\ i > 1 \end{cases} \\ 0 \end{cases}$$

** e^{e^x-1} $$f(i,j)a_i = -\frac{i}{i+j}\frac{1}{i!}$$

** $e^{\sin x}$ $$f(i,j)a_i = \begin{cases} -\dfrac{i}{i+j}\dfrac{(-1)^{\frac{i-1}{2}}}{i!} & \text{for } \begin{cases} i = 2k-1 \\ i = 2k \end{cases} \\ 0 \end{cases}$$

$$e^{\pm x} =$$

$$\left|\begin{matrix}
\frac{x^0}{0!} & \mp\binom{0}{0} & 0 & 0 & 0 & 0 & \rightarrow \\
\frac{x^1}{1!} & 1 & \mp\binom{1}{0} & 0 & 0 & 0 & 0 \\
\frac{x^2}{2!} & 0 & 1 & \mp\binom{2}{0} & 0 & 0 & 0 \\
\frac{x^3}{3!} & 0 & 0 & 1 & \mp\binom{3}{0} & 0 & 0 \\
\frac{x^4}{4!} & 0 & 0 & 0 & 1 & \mp\binom{4}{0} & 0 \\
\frac{x^5}{5!} & 0 & 0 & 0 & 0 & 1 & \mp\binom{5}{0} \\
\frac{x^6}{6!} & 0 & 0 & 0 & 0 & 0 & 1 \\
\downarrow & & & & & & \searrow
\end{matrix}\right|$$

$$e^{\pm e^x - 1} = \left| \begin{matrix} \frac{x^0}{0!} & \mp 1 & \mp 1 & \mp 1 & \mp 1 & \mp 1 & \mp 1 & \rightarrow \\ \frac{x^1}{1!} & 1 & \mp 1 & \mp 2 & \mp 3 & \mp 4 & \mp 5 & \\ \frac{x^2}{2!} & 0 & 1 & \mp 1 & \mp 3 & \mp 6 & \mp 10 & \\ \frac{x^3}{3!} & 0 & 0 & 1 & \mp 1 & \mp 4 & \mp 10 & \\ \frac{x^4}{4!} & 0 & 0 & 0 & 1 & \mp 1 & \mp 5 & \\ \frac{x^5}{5!} & 0 & 0 & 0 & 0 & 1 & \mp 1 & \\ \frac{x^6}{6!} & 0 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & & \searrow \end{matrix} \right|$$

Inside Pascal's triangle, compare with

$$\frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0} = \frac{1}{5!}\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix} \quad ; \quad \sum_{i=1}^{n} i^0 = n$$

determinant of 5th order

Pascal's triangle

$$1 + \ln\left(1 + \sum_{i=1}^{\infty} a_i x^i\right) =$$

$$\begin{vmatrix} x^0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & \rightarrow \\ x^1 & 1 & \frac{1}{2}a_1 & \frac{1}{3}a_2 & \frac{1}{4}a_3 & \frac{1}{5}a_4 & \frac{1}{6}a_5 & \\ x^2 & 0 & 1 & \frac{2}{3}a_1 & \frac{2}{4}a_2 & \frac{2}{5}a_3 & \frac{2}{6}a_4 & \\ x^3 & 0 & 0 & 1 & \frac{3}{4}a_1 & \frac{3}{5}a_2 & \frac{3}{6}a_3 & \\ x^4 & 0 & 0 & 0 & 1 & \frac{4}{5}a_1 & \frac{4}{6}a_2 & \\ x^5 & 0 & 0 & 0 & 0 & 1 & \frac{5}{6}a_1 & \\ x^6 & 0 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & & \searrow \end{vmatrix}$$

Find more!

F.e. sin(sinx)

sin(sinhx), sinh(sinx)

tan(tanx)

tan(tanhx), tanh(tanx)