Unit 30.1 – Mathematical analogues to maritime monster waves
Part 1 - Monster waves analogues in simple continued fractions
http://en.wikipedia.org/wiki/Monster_waves
http://de.wikipedia.org/wiki/Monsterwelle

30.1.1 Introduction

The simple continued fraction (scf)

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \quad \ldots + \cfrac{1}{a_n} = [a_0; a_1, a_2, a_3, \ldots, a_n]$$

$n \to \infty$ infinite scf

is a very efficient tool to analyze the "inner discrete (wave) structure" of mathematical constants and functions.

Algorithm: transformation $r \in \mathbb{R}_+ \to$ scf(r)

$r \in \mathbb{R}_+$; length(approx.scf(r))	
x:=r ; i:=0	
a_0:=int(x)	
repeat until	$x:=(x - \text{int}(x))^{-1}$
	i:=i+1
	a_i:=int(x)
x=int(x) or i=length	
n:=i	
approx.scf(r)=$[a_0;a_1,a_2,\ldots a_n]$	

f.e.
e = 2.7182818284…= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1,…]

But the scf representation is also a tool to model “monster waves” (huge signals; local-temporal energy monster impacts/lines/bullets) in a background of “calm sea” (small noise),
f.e.

$$\text{FibonacciBomb}(m,n) = \text{scf}\left[\left(\frac{\text{fib}(n+1)}{\text{fib}(n)}\right)^{m}\right] ;\ m,n \in \mathbb{N}$$

$$\text{fib}(0) = 0 \ ; \ \text{fib}(1) = \text{fib}(2) = 1$$

$$\text{fib}(n+2) = \text{fib}(n+1) + \text{fib}(n)$$

$$[1;\ 1,\ \underset{\text{n times}}{1, \ldots 1,}\ 1] = \frac{\text{fib}(n+1)}{\text{fin}(n)}$$

$$\lim_{n\to\infty} \frac{\text{fib}(n+1)}{\text{fib}(n)} = \frac{1}{2}(1+\sqrt{5}) = \Phi$$

$$\Phi^{m} = \text{fib}(m\text{-}1) + \text{fib}(m)*\Phi$$

$$\Phi^{1} = [1;1,1,1,1,1,1,1,1,1,1,1,1,1,\ldots]$$

$$\Phi^{2} = [2;1,1,1,1,1,1,1,1,,1,1,1,1,\ldots]$$

$$\Phi^{3} = [4;4,4,4,4,4,4,4,4,4,4,\ldots]$$

$$\Phi^{4} = [6;1,5,1,5,1,5,1,5,1,5,1,5,\ldots]$$

$$\Phi^{5} = [11,11,11,11,11,11,11,11,\ldots]$$

$$\Phi^{6} = [17;1,16,1,16,1,16,1,16,1,\ldots]$$

FibonacciBomb(5,50) =

[11, 11, 11, 11, 11, 11, 11, 11, 11, 13, 437845991669110338 04,
1, 5, 1, 10, 11, 11, 11, 11, 11, 11, 11, 11, 4, 1, 10, 11, 11, 11, 11,
11, 11, 11, 11, ...]

FibonacciBomb(5,55) =

[11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 12, 1,
538514970164685621520 1, 6, 1, 10, 11, 11, 11, 11, 11, 11, 11,
11, 11, 4, 1, 10, 11, 11, 11, 11, 11, 11, 11, 11, 11, ...]

FibonacciBomb(5,99) =

[11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11,
11, 11, 8, 1324677387061709725635162843114104137981 5, 1,
1, 5, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11,
11, 11, 11, 1, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11, 11,
11, 11, 11, 11, 11, 11, ...]

http://research.microsoft.com/en-us/um/people/cohn/contfrac.html
http://research.microsoft.com/en-us/um/people/cohn/Papers/monthly.pdf

Check in the same manner powers of scf segments of e, pi, and other mathematical constants.

General problem: how we can calm or suppress the "monsters" in all these and in the following scfs?
But also: how we can generate or reinforce "monsters in scf?

30.1.2 Infinite power tower series

The representation of the value of the infinite power tower series

$$\sum_{i=0}^{\infty} 3^{-3^i} = 0.370421175633926798495743189411268100978128596784605334815388602...$$

as a simple infinite continued fraction

$$\sum_{i=0}^{\infty} 3^{-3^i} = [0;2,1,2,3,26,1,2,2,1,2,19682,1,1,1,2,2,1,26,3,3,1,2,$$
$$7625597484986,1,1,1,2,3,26,1,2,2,1,1,1,19682,2,1,$$
$$2,2,1,26,3,2,1,2,$$
$$44342648824303776994824963061914989,1,1,1,$$
$$2,3,26,1,2,2,1,2,19682,1,1,1,2,2,1,26,3,2,1,1,1,$$
$$7625597484986,...]$$
$$= [0;2,1,(3^{3^0}-1),\ 3,(3^{3^1}-1),1,2,(3^{3^0}-1),\ 1,2,(3^{3^2}-1),$$
$$1,1,1,(3^{3^0}-1),2,1,(3^{3^1}-1),\ 3,(3^{3^0}-1),\ 1,2,(3^{3^3}-1),$$
$$1,1,1,(3^{3^0}-1),\ 3,(3^{3^1}-1),1,2,(3^{3^0}-1),1,1,1,(3^{3^2}-1),$$
$$2,1,(3^{3^0}-1),2,1,(3^{3^1}-1),\ 3,(3^{3^0}-1),\ 1,2,(3^{3^4}-1),$$
$$1,1,1,(3^{3^0}-1),\ 3,(3^{3^1}-1),1,2,(3^{3^0}-1),\ 1,2,(3^{3^2}-1),$$
$$1,1,1,(3^{3^0}-1),2,1,(3^{3^1}-1),\ 3,(3^{3^0}-1),1,1,1,(3^{3^3}-1),...]$$
$$= \left[0;a_i\right]_{i=1}^{\infty}$$
$$a_i \left\{ \begin{array}{c} = 3^{3^k}-1 \ \text{ for } i \bmod(6*2^k) = 3*2^k \ \underline{\text{or}}\ 3*2^k-1 \\ k=0,1,2,... \\ \in \{1,2.3\} \ \text{ else} \end{array} \right\}$$

shows exponentially increasing large “rogue waves” (large “signals”, large “impacts”) $(3^{3^k}-1)$; k=1,2,3,... embedded in a background of “small waves” (“noise” background) 1,2,3.

Extension 1

$$f(3,x\in\mathbb{N})=\sum_{i=0}^{\infty} x^{-3^i} =$$

$$[a_0;x\text{-}1,1,(x^{3^0}-1),\quad x,(x^{3^1}-1),1,x\text{-}1,(x^{3^0}-1),\quad 1,x\text{-}1,(x^{3^2}-1),$$
$$1,x\text{-}2,1,(x^{3^0}-1),\ x\text{-}1,1,(x^{3^1}-1),\quad x,(x^{3^0}-1),\quad 1,x\text{-}1,(x^{3^3}-1),$$
$$1,x\text{-}2,1,(x^{3^0}-1),\quad x,(x^{3^1}-1),1,x\text{-}1,(x^{3^0}-1),1,x\text{-}2,1,(x^{3^2}-1),$$
$$x\text{-}1,1,(x^{3^0}-1),\ x\text{-}1,1,(x^{3^1}-1),\quad x,(x^{3^0}-1),\quad 1,x\text{-}1,(x^{3^4}-1),$$
$$1,x\text{-}2,1,(x^{3^0}-1),\quad x,(x^{3^1}-1),1,x\text{-}1,(x^{3^0}-1),\quad 1,x\text{-}1,(x^{3^2}-1),$$
$$1,x\text{-}2,1,(x^{3^0}-1),\ x\text{-}1,1,(x^{3^1}-1),\quad x,(x^{3^0}-1),1,x\text{-}2,1,(x^{3^3}-1),\ldots]$$

$$= \left[a_0;a_i\right]_{i=1}^{\infty}$$

$$a_i\begin{cases} = x^{3^k}-1 \ \text{ for } i \bmod(6*2^k)=3*2^k \ \underline{\text{or}}\ 3*2^k-1 \\ k=0,1,2,\ldots \\ \in\{1,x\text{-}2,x\text{-}1,x\} \ \text{ else} \end{cases}$$

Extension 2

Find a similar simple infinite continued fraction representation

for $f(p,p) = \sum_{i=0}^{\infty} p^{-p^i}$; p = prime , esp. for p = 5, 7, 11

For comparison

$$\sum_{i=1}^{\infty} 2^{-2^i} = [0;3,6,4,4,2,4,6,4,2,6,4,2,4,4,6,A_1,A_1^*,$$

$$6,4,4,2,4,6,2,4,6,4,2,4,4,6,A_2,A_2^*,$$

$$\ldots \quad ,4,2, \quad \ldots \quad ,A_3,A_3,$$

$$\ldots \quad ,2,4, \quad \ldots \quad ,A_4,A_4^*,$$

$$\ldots \quad ,4,2, \quad \ldots \quad ,A_5,A_5^*,$$

$$\ldots \quad ,2,4, \quad \ldots \quad ,A_6,A_6^*,\ldots]$$

with

$$A_i = \begin{cases} 4 & i \bmod 4 = 1 \\ 4 & i \bmod 8 = 2 \\ 2 \quad \text{for} & i \bmod 4 = 3 \\ 2 & i \bmod 8 = 6 \\ 4 & i \bmod 4 = 0 \end{cases}$$

$$A_i^* = 2 + A_i \bmod 4$$

30.1.3

A prime asymmetry: $p_-=6k-1$ and $p_+=6k+1$ primes generate different kinds of continued fractions containing monsters

All primes $\geqslant 5$ have the form

$$p_\pm = 6k \pm 1;\ k \in \mathbb{N}. \qquad (1)$$

The infinite simple continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} = [a_0\,;\ a_1,\, a_2,\, a_3,\, ...]\,;\ a_i \in \mathbb{N}_0 \tag{2}$$

of the prime function

$$f(p) = \frac{\sqrt[p]{e}}{3 - 2\sqrt[p]{e}} \tag{3}$$

are sequences of triplets of triplets both for p_ and p_+:

$$f(p_-) = [\ \underbrace{\underbrace{1\,;\ c_0,\, 1}_{\text{elementary triplet}}\ ,\ \underbrace{5,\, c_1,\, 1}\ ,\ \underbrace{5,\, c_2,\, 5}}_{\text{main triplet}}\ ,\ \underbrace{\underbrace{1,\, c_3,\, 1}_{\text{elementary triplet}}\ ,\ \underbrace{5,\, c_4,\, 1}\ ,\ \underbrace{5,\, c_5,\, 5}}_{\text{main triplet}}\ ,\ ...] \tag{4}$$

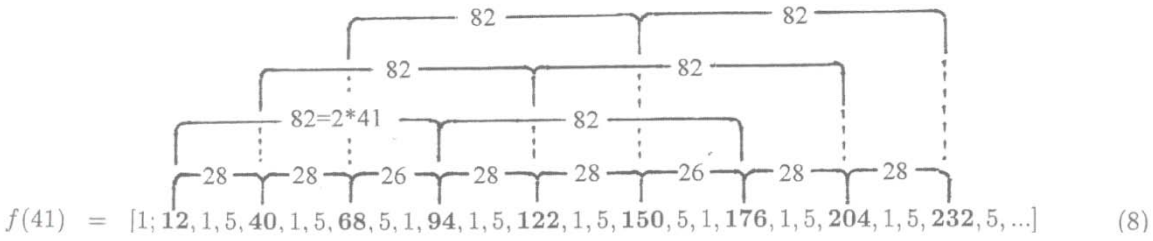
$$f(p_+) = [\underbrace{1\,;\ d_0,\, 1}, \underbrace{1,\, d_1,\, 1}, \underbrace{1,\, d_2,\, 5}, \underbrace{1,\, d_3,\, 1}, \underbrace{1,\, d_4,\, 1}, \underbrace{1,\, d_5,\, 5}, ...]\,, \tag{5}$$

but with different arrangements of the bodyguard elements (noise) 1 and 5 of the elementary triplets and with remarkably different structures of the variable elements (signals c_i and d_i, respectively:

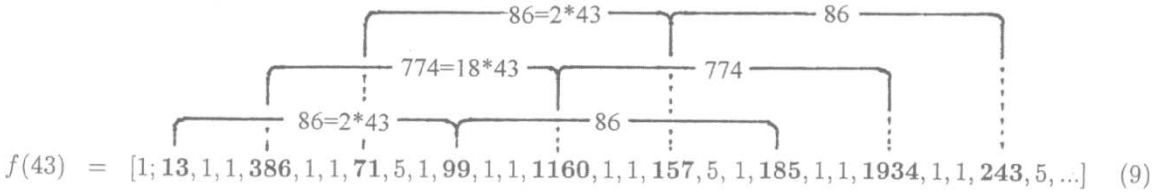
$$\begin{aligned} c_i &= \frac{p_- + 1}{3}\Big(2\,i + 1\Big) - 2\Big(\text{int}\Big(\frac{i}{3}\Big) + 1\Big) \\ &= 2k\Big(2\,i + 1\Big) - 2\Big(\text{int}\Big(\frac{i}{3}\Big) + 1\Big) \\ &= c_0 + (c_1 - c_0)i - 2\text{int}\Big(\frac{i}{3}\Big);\ c_0 = \frac{p_- - 5}{3} = 2(k-1)\,;\ c_1 = p_- - 1 = 2(3k-1) \end{aligned} \tag{6}$$

$$d_i = \begin{cases} d_0 + (d_3 - d_0)\frac{i}{3} & i\,\text{mod}3 = 0 \\ d_1 + (d_4 - d_1)\frac{i-1}{3} \quad \text{for} & i\,\text{mod}3 = 1 \\ d_2 + (d_5 - d_2)\frac{i-2}{3} & i\,\text{mod}3 = 2 \end{cases}\ ; \quad \begin{array}{c|c} i & d_i \\ \hline 0 & 2k-1 \\ 1 & 54k+8 \\ 2 & 10k+1 \\ 3 & 14k+1 \\ 4 & 162k+26 \\ 5 & 22k+3 \end{array} \quad ;\ 2k = \frac{p_+ - 1}{3} \tag{7}$$

So we have f.i. for the prime twins p_=41 and p_+=43 (k=7):



$$f(41) = [1; \mathbf{12}, 1, 5, \mathbf{40}, 1, 5, \mathbf{68}, 5, 1, \mathbf{94}, 1, 5, \mathbf{122}, 1, 5, \mathbf{150}, 5, 1, \mathbf{176}, 1, 5, \mathbf{204}, 1, 5, \mathbf{232}, 5, \ldots] \quad (8)$$



$$f(43) = [1; \mathbf{13}, 1, 1, \mathbf{386}, 1, 1, \mathbf{71}, 5, 1, \mathbf{99}, 1, 1, \mathbf{1160}, 1, 1, \mathbf{157}, 5, 1, \mathbf{185}, 1, 1, \mathbf{1934}, 1, 1, \mathbf{243}, 5, \ldots] \quad (9)$$

The continued fractions (8) and (9) illustrate the aforementioned difference between c_i and d_i. In the first case p_ have all successive c_i-differences inside a main triplet and all differences (c_{i+3}-c_i) the same values:

$$(\forall i)_{i \bmod 3=0}\ (c_{i+1} - c_i) = (\forall i)_{i \bmod 3=1}\ (c_{i+1} - c_i) = \frac{2}{3}\left(p_- + 1\right) \quad (10)$$

$$(\forall i)\ (c_{i+3} - c_i) = 2\,p_- \,. \quad (11)$$

In the case of p_+ the structure of the accompanying continued fraction (5) is more difficult:

$$(\forall i)_{i \bmod 3=0}\ (d_{i+1} - d_i) \neq (\forall i)_{i \bmod 3=1}\ (d_{i+1} - d_i) \quad (12)$$

$$(\forall i)_{i \bmod 3=0}\ (d_{i+3} - d_i) = (\forall i)_{i \bmod 3=2}\ (d_{i+3} - d_i) = 2\,p_+ \quad (13)$$

$$(\forall i)_{i \bmod 3=1}\ (d_{i+3} - d_i) = 18\,p_+ \,. \quad (14)$$

Discuss applications, esp. of this prime asymmetry.

3.1.4
Champernowne constant scf monster wave

http://mathworld.wolfram.com/ChampernowneConstant.html

C_{10} = 0.12345678910111213141516171819
20212223242526272829303132333435…

The simple continued fraction of the Champernowne constant is [0; 8, 9, 1, 149083, 1, 1, 1, 4, 1, 1, 1, 3, 4, 1, 1, 1, 15,
457540 11139103107648364662824295611859960393971045755500066200439309026265925631493795320774712865631386412093755035520946071 8

308998457580146986314883359214 1783010987, 6, 1, 1, 21, 1, 9, 1, 1, 2, 3, 1, 7, 2, 1, 83, 1, 156, 4, 58, 8, 54, ...]

3.1.5 Miscellaneous

3.1.5.1

$$\frac{2393+144*\Phi}{1479+89*\Phi}=1.617984451...$$

= [1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 17, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1, 1,…]

This example shows “a **Rogue Wave** of the high 17 in the middle of normal sea waves of the high 1”. The next example models another form of monster waves, the “**Three Sisters** 17, 17, 17 among normal sea waves 1”.

$$\frac{2405+41026*\Phi}{1501+25605*\Phi}=1.602265182...$$

= [1; 1, 1, 1, 1, 17, 17, 17, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1, 1, 1,…]

We have here a tool to model different kind of monster waves in a strong nonlinear sea system with random influences as a possible approach to a deeper understanding of their formation and maybe to their prediction. Such monster waves can work havoc on ships, offshore oil platform, offshore wind parks, and coasts.
Monster wave → ? local-temporal energy monster impact/line/bullet in a low energy background?

3.1.5.2

Partition of a power tower

$$
\begin{aligned}
4^{3^{2^1}}+1 &= 262145 \\
&= (3^0\cdot 4+1)(3^1\cdot 4+1)(3^2\cdot 4+1)(3^3\cdot 4+1) \\
&\qquad\quad \| \qquad\quad \| \qquad\quad \| \qquad\quad \| \\
&\qquad 5\in\mathbb{P} \qquad 13\in\mathbb{P} \qquad 37\in\mathbb{P} \qquad 109\in\mathbb{P}
\end{aligned}
$$

$$
\underbrace{\qquad\qquad}_{4^1+1}
$$

$$
\underbrace{\qquad\qquad\qquad\qquad}_{4^3+1}
$$

$$
\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{4^{3^{2^1}}+1}
$$

Find other power towers of this kind.