Unit 23 Pick variations

“The core of the creativity is the ability to perceive something in a new way.

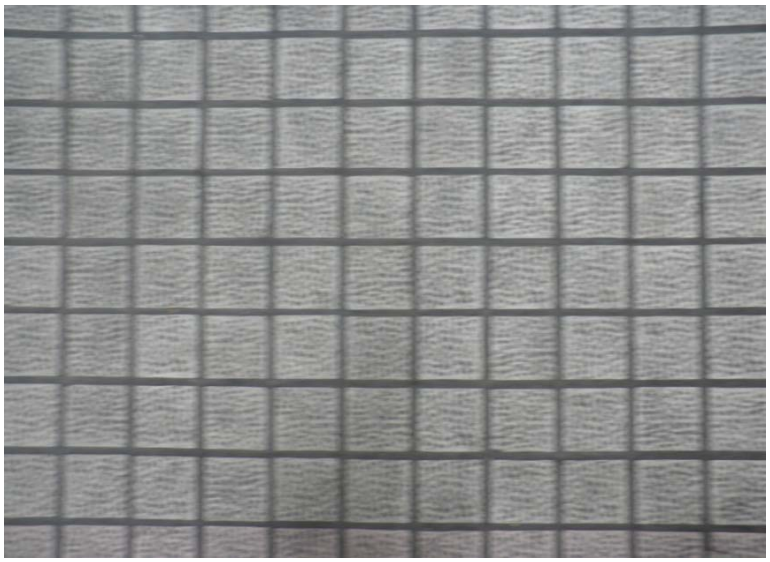
Kern der Kreativität bildet die Fähigkeit, etwas neu wahrzunehmen.”

D.R. Hofstädter und die Fluid Analogies Group:
Die FARGonauten. Über Analogie und Kreativität.
Stuttgart 1996

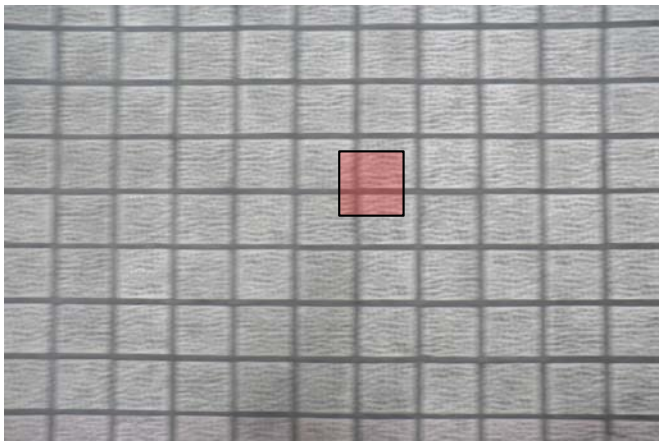
Kreatives Denken, etwas neu wahrnehmen bedeutet hier:
analytisch sehen, ein vorliegendes Problem analysieren:
* Welche Elemente?
* Welche Verbindungen existieren zwischen den Elementen?
* Ist alles sichtbar („verborgene Parameter“)?

Creative thinking, to perceive (to see) something in a new way (under a new viewpoint) means here to perceive analytically, to analyze a given problem with penetrating clarity:
* Which elements?
* Which connections exist between the elements?
* Is all visible (hidden parameters)?

Ein einfaches Beispiel. A simple example.
Eine Fingerübung in Kreativität. A creative fingerexercise.
Im CQU Campus A sehen wir an mehreren Stellen eine Pflasterung mit einfachen quadratischen Kacheln, ein einfaches quadratisches Gitter mit der Seitenlänge a.
In CQU campus A, we see at several places a paving with simple square tiles, a 2D quadratic lattice with side length a.
Wir sehen es jeden Tag und gehen daran vorüber.
We see it every day and pass it by.

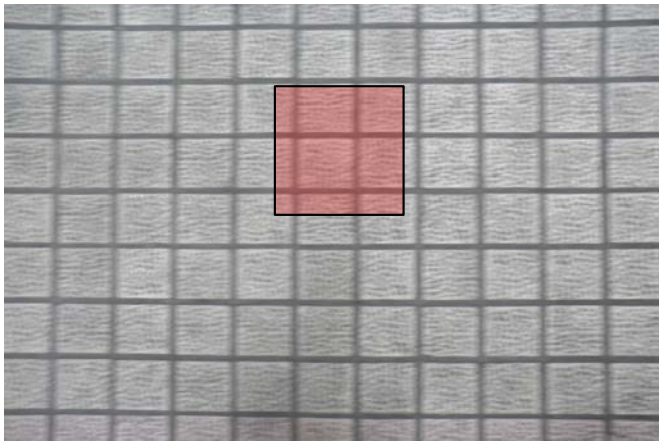


Heute bleibe ich zum ersten Male stehen und ich färbe in Gedanken eine Kachel (= ein Quadrat mit der Seitenlänge a) rot.
Today, I stop at first time and I color in thoughts one tile (= a square with side length a) red.



I = 0
B = 4
S = 1

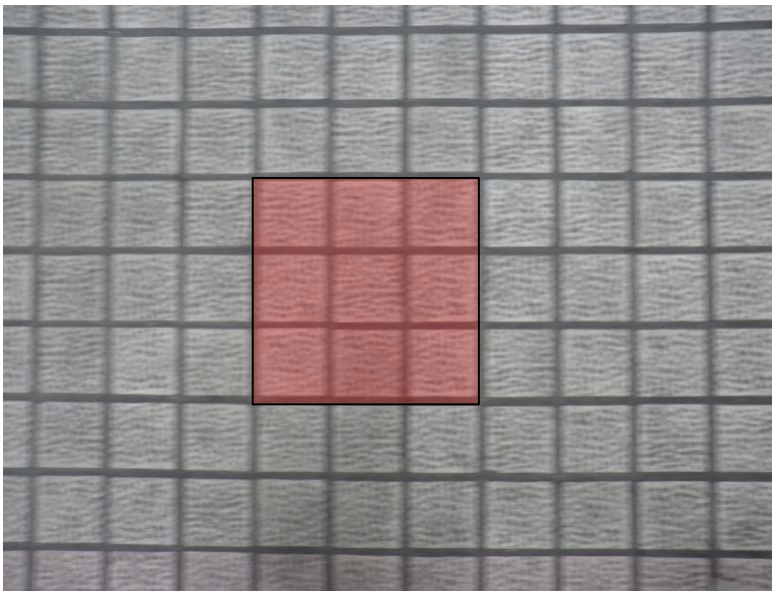
Danach bilde ich in Gedanken aus 4 benachbarten Kacheln ein rotes Quadrat mit der Seitenlänge 2a.
After it, I form in my imagination from 4 neighboring tiles a red square with side length 2a.



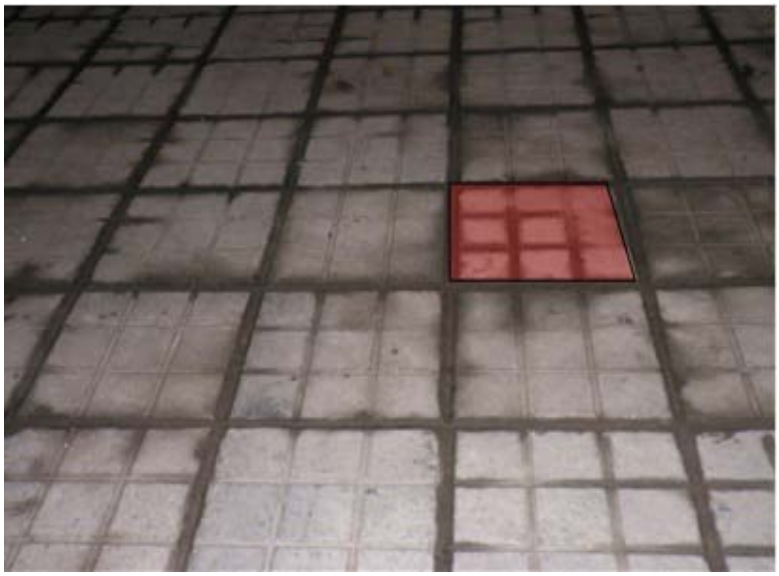
I = 1
B = 8
S = 4

Das wiederhole nun mit 9 (i2) benachbarten Kacheln = rotes Quadrat mit der Seitenlänge 3a (ia), und so fort: i = 4, 5, 6, ...

I repeat this now with 9 (i2) neighboring tiles = red square with side length 3a (ia), and so on: i = 4, 5, 6, ...



I = 4
B = 12
S = 9



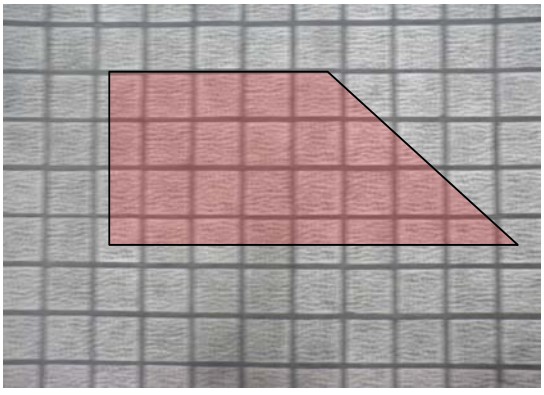
I = 4
B = 12
S = 9

Welche numerischen Elemente habe ich hier in/auf einem roten Quadrat mit der Seitenlänge ia?
Which numerical elements in/on a red square with side length ia?
* S(i) –Anzahl der Kacheln, number of tiles
* I(i) – Anzahl der inneren Gitterpunkte, inner lattice points
* B(i) – Anzahl der Randgitterpunkte, boundary lattice points
Welche Beziehungen bestehen nun zwischen den Elementen?
Which connections exist between the elements?
* S(i) = i2
* I(i) = S(i-1) = (i-1)2
* B(i) = 2(i+1) + 2(i-1) = 4i ,
in Tabellenform, as table

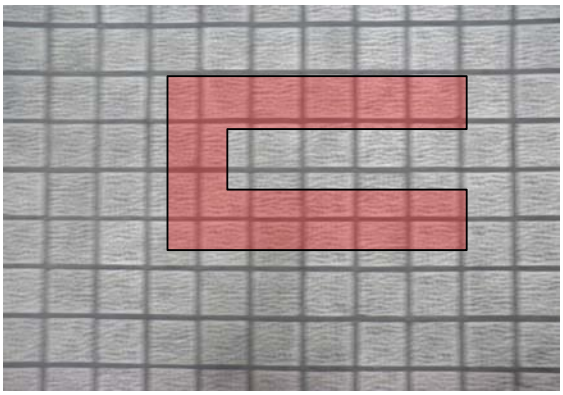
i	S(i)	I(i)	S(i) – I(i)	B(i)
1	1	0	1	4
2	4	1	3	8
3	9	4	5	12
4	16	9	7	16
5	25	16	9	20
i	i2	(i-1)2	2i-1	4i

Der Flächeninhalt A(i) des roten Quadrates ist dann
The area A(i) of the red square is given by
A(i) = S(i)a2 .
Aus der Tabelle erkenne ich unmittelbar, dass zwischen S, I und B die einfache lineare Beziehung S – I = ½B -1, S = I + ½B -1 besteht.
From the table I recognize immediately the simple linear

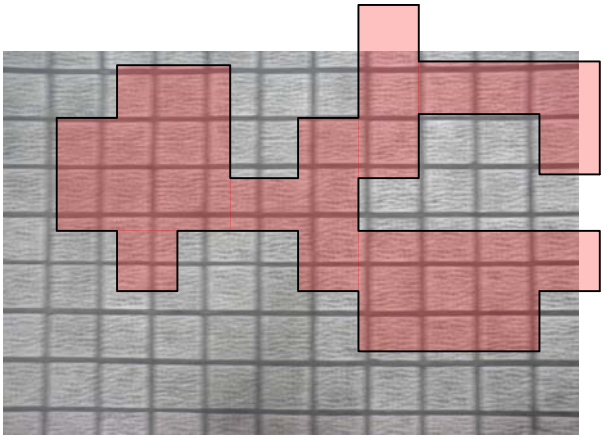
connection S – I = ½B -1, S = I + ½B -1 .
Das bedeutet aber: That means:
Der Flächeninhalt eines Quadrates in einem quadratischen Punktgitter mit der Seitenlänge a läßt sich allein durch Auszählen seiner inneren und äußeren Gitterpunkte berechnen:
We can calculate the area of a square in a quadratic lattice with side length a only by counting of its inner and boundary lattice points:
$A = (I + ½B – 1)a^2$ (SP)
Das ist ein erstaunliches Ergebnis. Ich habe es allein durch „genaueres Sehen“ (dazu gehört auch die mentale analytische Visualisierung des Gesehenen auf dem „inneren Farbdisplay“) und darauf basierendes analytisches Denken erhalten.
This is a remarkable result, I found it alone by „analytic seeing” and creative thinking. And it is extensible!
Noch erstaunlicher aber ist, dass diese Berechnung des Flächeninhaltes auf ein beliebiges einfaches n-Eck (mit konvexen und konkaven Ecken, aber keine Löcher) in einem quadratischen Punktgitter mit der Seitenlänge a erweitert werden kann: Satz von Pick.
http://de.wikipedia.org/wiki/Satz_von_Pick
http://en.wikipedia.org/wiki/Pick%27s_theorem



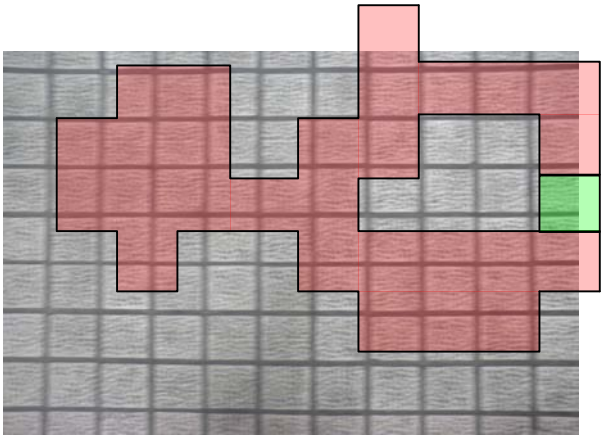
I = 9
B = 17
S = 16.5
S – (I +½B)
+1= 0



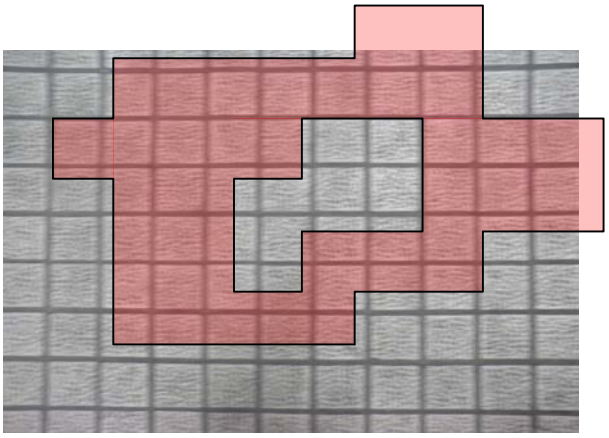
I = 0
B = 24
S = 11
S – (I + ½B)
+1 = 0



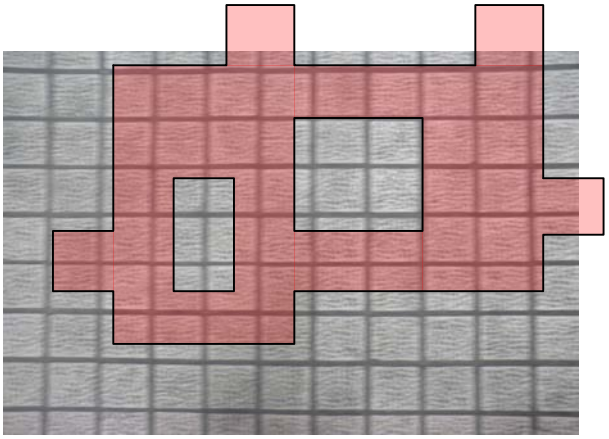
I = 5
B = 46
S = 27
S – (I +½B)
+1=0



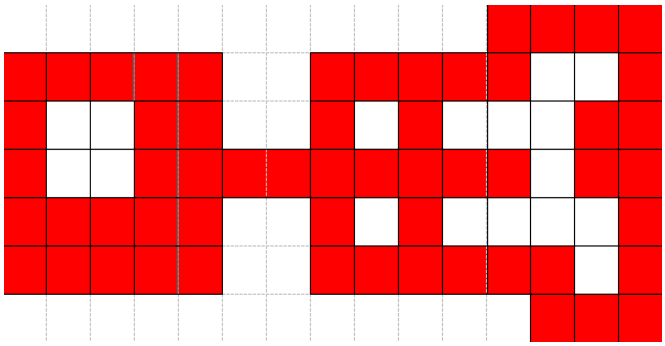
I = 5
B = 46
S = 27+1=
28
S – (I + ½B)
+1= 1



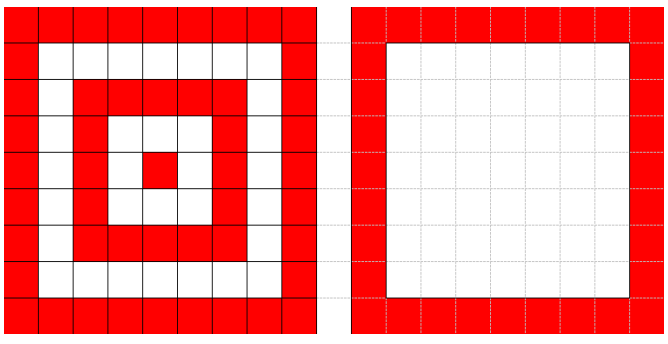
I = 8
B = 30+12 = 42
S = 29
S – (I + ½B) +1= 1



I = 5
B = 32+6+46
S = 29
S – (I + ½B) +1= 2



I = 8
B = 52+8+4+24= 92
S = 57
S – (I + ½B) +1= 4

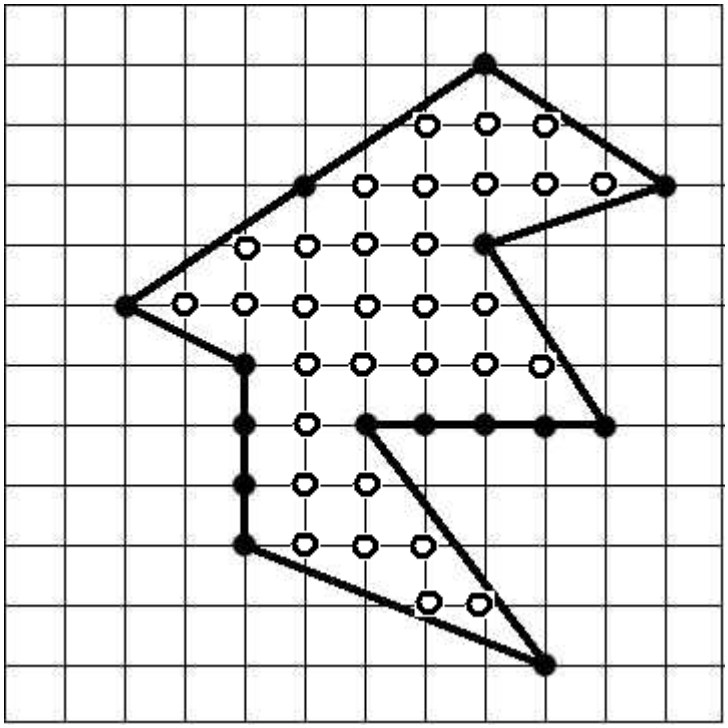


I = 0
B = 64
S = 32
S – (I + ½B) +1= 1

Discuss this case!

Conjecture → "experimental" and mathematical proof:
S – (I + ½B) +1 = number of holes in a "□shell" .
This magic formula can abstract from the holes (shape, area, position) to the number of holes.

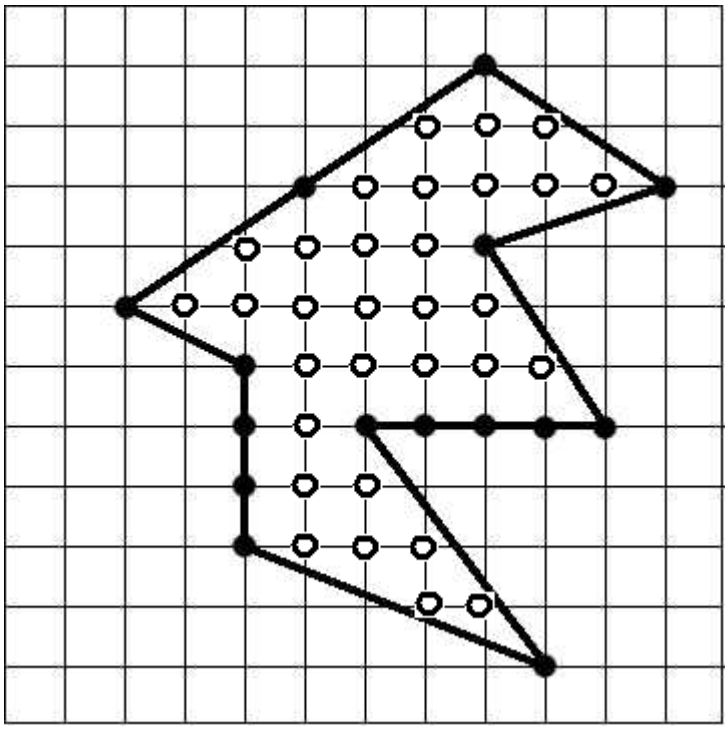
Now extension from □structure to a free polygon, all vertices are lattice points:



I = 31
B = 15
S = 37.5
S – I – 0.5B = -1

How about the holes in this case?

Another approach
Given a simple polygon. Its vertices are lattice points in a twodimensional lattice.
Find a simple formula expression for the area of the polygon in function of the number I of the interior lattice points (o) and the number B of the boundary lattice points (•) of the polygon.
Go the simplest way.



I = 31
B = 15
S = 37.5
A =
37.5a^2

Solution:
Simplest attempt: It is obvious that S growths linearly with I, but the boundary points (B) play also a role, therefore we use a linear approach (a linear combination in I and B)
S= aI + bB + c. (120)
Now we have to determine the values of the three unknown constants a, b and c.
If equ. (120) is true, then it must be true also for three very simple special cases:
1. • • I = 0, B = 4, S = 1
   • •
2. • • • • I = 0, B = 8, S = 3
   • • • •
3. • • • • I = 2, B = 10, S = 6
   • o o •
   • • • •

From 1., 2. and 3. follows
4b + c = 1
8b + c = 3
2a +10b + c = 6 .
And finally
a = 1, b = ½, c = -1 .
Result

$S = I + \frac{1}{2}B - 1$ Pick's Theorem (PT)

Now: experimental test and mathematical proof of equ. (PT)!

See

http://www.mcs.drexel.edu/~crorres/Archimedes/Stomachion/Pick.html

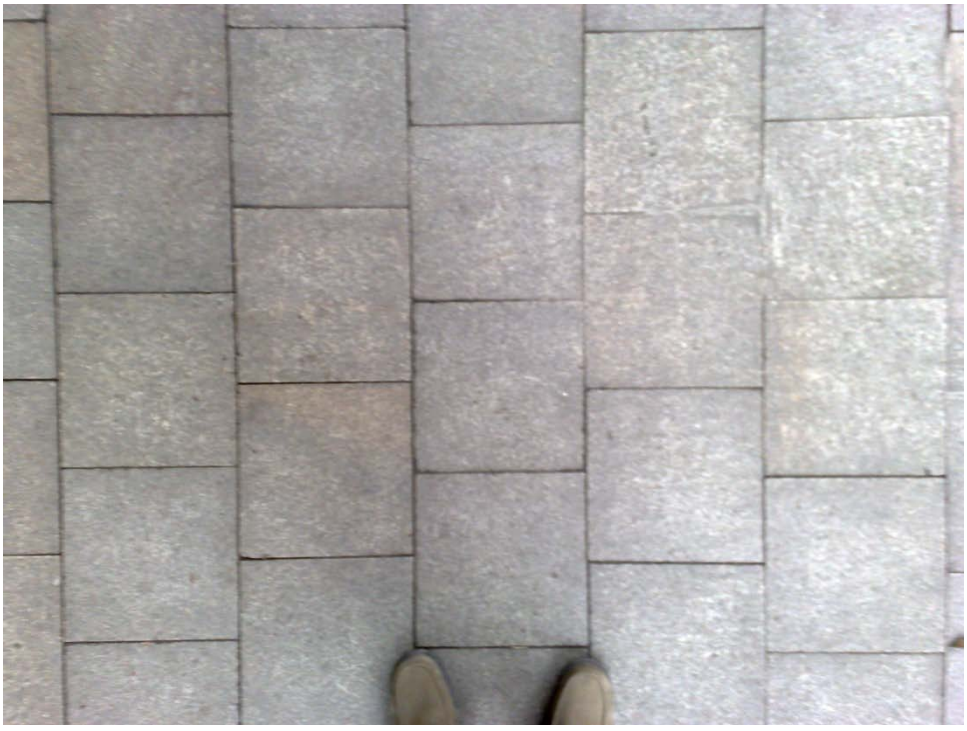
http://episte.math.ntu.edu.tw/articles/sm/sm_25_10_1/index.html

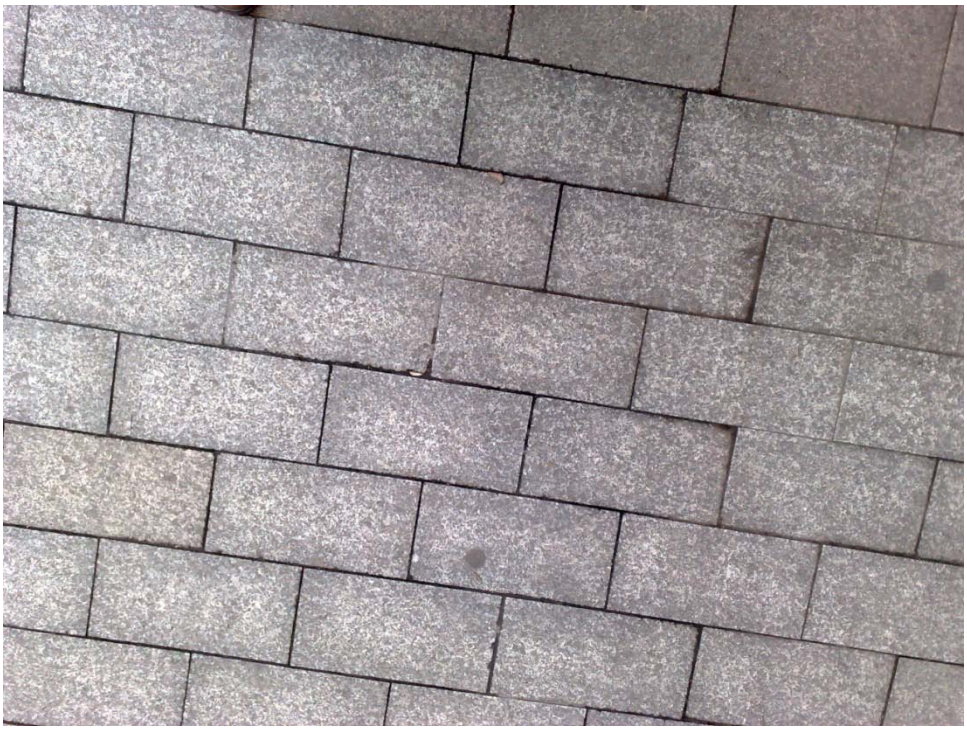
Problems:

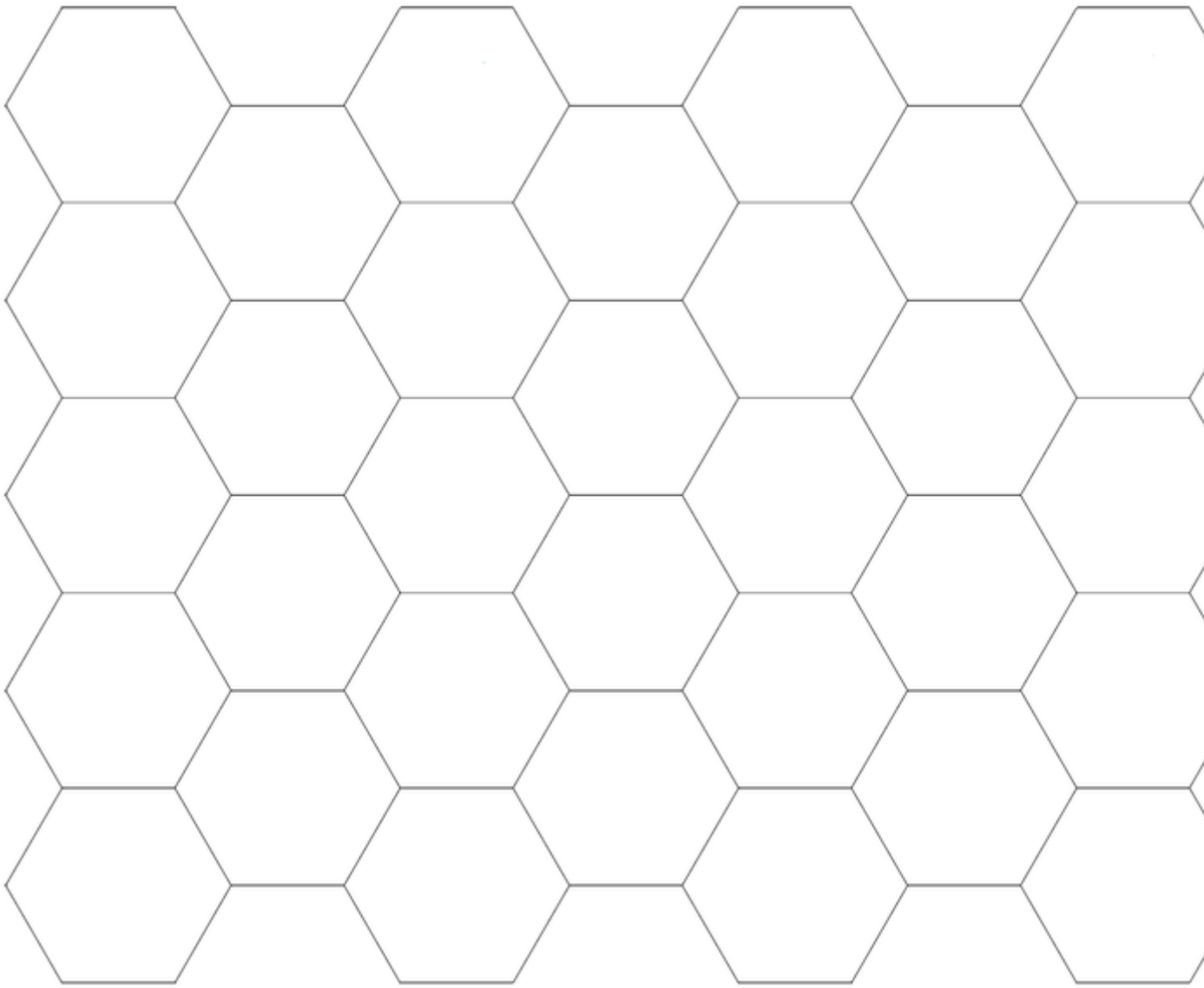
* How about other parameters of the polygon (perimeter, centroid = mass center, momentum of inertia, etc.) in function of I and B?
  * Gibt es „Negativfälle" beim Pick's Theorem?
  * How about equ. (PT) if a polygonal hole is inside the polygon?
* How about a 3D extension of Pick's Theorem (volume, surface)? Start with a simple special case: x-y-z-axis parallel located cube.
  See http://www.ics.uci.edu/~eppstein/junkyard/pick3d.html
* Discuss a connection between Pick's Theorem and Euler's polyhedron formula, see
  http://www.ics.uci.edu/~eppstein/junkyard/euler/pick.html
  http://de.wikipedia.org/wiki/Eulersche_Polyederformel
* Application fields of Pick's Theorem and its extensions: image and pattern processing, and ?engineering?

* Läßt sich der Satz von Pick auch auf andere „Parkettierungen" (geometric patterns from tiles and brickwork) uebertragen, z.B.









Hexagonal tiling: $2S - (I + \frac{1}{2}B) + 1$ = number of holes ?
→ experimental and mathematical proof.

S – number of grid elements (tiles: triangles, squares, hexagons)
I – number of inner lattice points
B – number of boundary lattice points of this polygon.
Extension of Pick's theorem
$aS - (I + \frac{1}{2}B) + 1$ = number of holes

grid element	a
triangle	½
square	1
hexagon	2
dodecahedron	3/2

More:

- R. Field. Geometric Patterns from Tiles & Brickwork.
  Tarquin Publications 2004, ISBN 1-899618-12-0
  in Chinese ISBN 7-5777-0415-3
- C. Martin. Weaving. Methods, Patterns and Traditions of an Ancient Art.
  Wooden Books 2007, ISBN 1-904263-55-0
- P.J. Lu, P.J. Steinhardt. Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture.
  Science vol. 315, no. 5815, p.1106-1110
  The wonderful complex girih (geometric star-and-polygon) patterns in medieval Islamic architecture were created as tessellations of a set of only five girih tiles: decagon, pentagon, hexagon, rhombus, and bowtie.
- http://mathworld.wolfram.com/Tiling.html
  http://www.primini.homepage.t-online.de/parkettierung-7.html
  http://mathworld.wolfram.com/Tiling.html
  http://mathworld.wolfram.com/PenroseTiles.html
  http://mathworld.wolfram.com/AnisohedralTiling.html

* Extremste Verallgemeinerung (Parallelperspektive, Projektionen)? Relation zwischen Anzahl der Elementarflächen, Anzahl der inneren Gitterpunkte und Anzahl der Randgitterpunkte?