Unit 20
An extension of the differential calculus quotient rule and beyond

Talk at the Sichuan University, Chengdu
Mathematical Department
Yangtze Center of Mathematics

The talk - part of a lecture course on Algorithm Engineering -presents an application-oriented attempt of a **synopsis** of the dc product, **quotient**, and chain **rule extension**s
based on
* the **normalized** differential operator notation

$$D^{(i)} = \frac{d^i}{i!dx^i} \; ; \; D^{(i)}f(x) = F^{(i)}$$

* a resulting two-dimensional formula representation by matrices and determinants **without numeric coefficients**

including
* Faa di Bruno's formula,
* an extended Newton's root finder,
* Taylor series representation of fractional functions,
* solution of linear differential equations, esp. applications in electrical engineering,
* a zoo of reciprocals of the factorial determinant patterns,
* sum of powers of natural numbers,
* polynomial handling, esp. a rational roots finder (wormholes through the irrationality).

A **prospective application area** in physics and engineering is the systematic design of static and dynamic, stationary and movable discrete structures which are able to guide a continuous wave or dynamic field around an object without considerable energetic interaction (esp. metastructures with negative refractive index; 2D $\rightarrow$ 3D; macro, micro, nano range).

1. Introduction
2. From the normalized Leibniz identity to the extended quotient rule

3. In the shadow of the determinant pattern of the extended quotient rule
3.1 Faa di Bruno's formula
3.2 Newton's root finding algorithm and more
3.3 How about the special cases u(x) = v'(x) and v(x) = u'(x) ?
3.4 Taylor series representation of fractional functions
$f(x) = \frac{u(x)}{v(x)}$ at a=0
4. Solution of linear differential equations, esp. applications in electrical engineering
5. A zoo of reciprocals of the factorial determinant patterns
5.1 Connections with Bernoulli and Euler numbers
5.2 A connection with π
5.3 A connection with e
5.4 Connections with other mathematical constants
5.5 Connections with primes
5.6 Miscellaneous
6. Sum of powers of natural numbers
6.1 Sum
6.1.1 Sum of even powers
6.1.2 Sum of odd powers
6.2 ±oscillating sum
6.2.1 ±oscillating sum of even powers
6.2.2 ±oscillating sum of odd powers
6.3 Other oscillation patterns sum
7. Extended quotient rule in an algorithmic-oriented representation (array + navigation rule + local operator)
8. Applications

1. Introduction

In the textbooks on differential calculus we can find three important derivative rules:

* **product rule**

$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$ (1)

and its extensions

* Leibniz identity

$$(u(x)v(x))^n = \sum_{i=0}^{n} \binom{n}{i} u^{(i)}(x)v^{(n-i)}(x) \qquad (2)$$

* $(u(x)v(x)w(x))' = u'(x)v(x)w(x) + u(x)v'(x)w(x) + u(x)v(x)w'(x)$ (3)

* **quotient rule**

* $$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} \qquad (4.1)$$

* normalized

$$\frac{\left(\frac{u(x)}{v(x)}\right)'}{\left(\frac{1}{v(x)}\right)'} = u(x) - v(x)\frac{u'(x)}{v'(x)} \qquad (4.2)$$

* **chain rule**

$(f(v(x)))' = (fov)' = f'(v(x))v'(x)$ (5.1)

and its extension

* Faa di Bruno's formula, first derivatives

$(f(v(x))' = f\,'v'$

$(f(v(x))'' = f\,'v'' + f\,''v'v'$

$(f(v(x))''' = f\,'v''' + 3f\,''v'v'' + f'''v'v'v'$

$(f(v(x))'''' = f\,'v'''' +4f\,''v'v''' + 3f\,''v''v'' + 6f\,'''v'v'v'' + f\,''''v'v'v'v'$ (5.1-5)

$f^{(i)} = f^{(i)}(v(x))\ ;\ v^{(i)} = v^{(i)}(x)$

All this rules and their extensions are very important mathematical tools in physics and engineering .

In the following I want to present you an application-oriented attempt of a synopsis of these rules and extensions based on

* the **normalized** differential operator notation

$$D^{(i)} = \frac{d^i}{i!dx^i}\ ;\ D^{(i)}f(x) = F^{(i)}\ ;\ D^{(i)}u(x) = U^{(i)}\ ;\ D^{(i)}v(x) = V^{(i)} \qquad (6)$$

* a resulting two-dimensional formula representation by matrices and determinants **without numeric coefficients**.

2. From the normalized Leibniz identity to the extended quotient rule

With

f (x) = u(x)v(x) ; F = UV (7.1)

and equ. (6) we obtain the normalized Leibniz identity in string representation

$$F^{(n)} = (UV)^{(n)} = \sum_{i=0}^{n} U^{(i)} V^{(n-i)} = \sum_{i=0}^{n} V^{(n-i)} U^{(i)}, \qquad (7.2)$$

and in matrix representation

$$\underbrace{\begin{bmatrix} F \\ F' \\ F'' \\ F''' \\ F^{(4)} \\ \vdots \end{bmatrix}}_{\vec{F}} = \begin{bmatrix} (UV) \\ (UV)' \\ (UV)'' \\ (UV)''' \\ (UV)^{(4)} \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} V & 0 & 0 & 0 & 0 & \\ V' & V & 0 & 0 & 0 & \\ V'' & V' & V & 0 & 0 & \\ V''' & V'' & V' & V & 0 & \\ V^{(4)} & V''' & V'' & V' & V & \\ & & \vdots & & & \ddots \end{bmatrix}}_{\underline{V}} \underbrace{\begin{bmatrix} U \\ U' \\ U'' \\ U''' \\ U^{(4)} \\ \vdots \end{bmatrix}}_{\vec{U}} \qquad (8)$$

The solution of equ. (8) for $\vec{U}$ and $U^{(4)}$, respectively, gives

$$\begin{bmatrix} U \\ U' \\ U'' \\ U''' \\ U^{(4)} \\ \vdots \end{bmatrix} = \begin{bmatrix} V & 0 & 0 & 0 & 0 & \\ V' & V & 0 & 0 & 0 & \\ V'' & V' & V & 0 & 0 & \\ V''' & V'' & V' & V & 0 & \\ V^{(4)} & V''' & V'' & V' & V & \\ & & \vdots & & & \ddots \end{bmatrix}^{-1} \begin{bmatrix} F \\ F' \\ F'' \\ F''' \\ F^{(4)} \\ \vdots \end{bmatrix} \qquad (9)$$

and via Cramer's rule for n=4

$$U^{(4)} = \frac{1}{V^{4+1}} \begin{vmatrix} V & 0 & 0 & 0 & F \\ V' & V & 0 & 0 & F' \\ V'' & V' & V & 0 & F'' \\ V''' & V'' & V' & V & F''' \\ V^{(4)} & V''' & V'' & V' & F^{(4)} \end{vmatrix}$$

$$= \frac{(-1)^4}{V^{4+1}} \begin{vmatrix} F & V & 0 & 0 & 0 \\ F' & V' & V & 0 & 0 \\ F'' & V'' & V' & V & 0 \\ F''' & V''' & V'' & V' & V \\ F^{(4)} & V^{(4)} & V''' & V'' & V' \end{vmatrix} \qquad (10)$$

Now we change in equ. (10) U↔ F to

$$F^{(4)} = \left(\frac{U}{V}\right)^{(4)} = \frac{(-1)^4}{V^{4+1}} \begin{vmatrix} U & V & 0 & 0 & 0 \\ U' & V' & V & 0 & 0 \\ U'' & V'' & V' & V & 0 \\ U''' & V''' & V'' & V' & V \\ U^{(4)} & V^{(4)} & V''' & V'' & V' \end{vmatrix} \qquad (11)$$

The heuristic extension 4 →n gives finally the normalized **extended quotient rule**

$$F^{(n)} = \left(\frac{U}{V}\right)^{(n)} = \frac{(-1)^n}{V^{n+1}} \underbrace{\begin{vmatrix} U & V & 0 & 0 & \cdots & 0 & 0 \\ U' & V' & V & 0 & \cdots & 0 & 0 \\ U'' & V'' & V' & V & & 0 & 0 \\ U''' & V''' & V'' & V' & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ U^{(n-1)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & \cdots & V' & V \\ U^{(n)} & V^{(n)} & V^{(n-1)} & V^{(n-2)} & \cdots & V'' & V' \end{vmatrix}}$$

(n+1)×(n+1) determinant

(12.1)

3. In the shadow of the determinant pattern of the extended quotient rule equ. (12.1)

$$\frac{d^n\left(\frac{u(x)}{v(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{u(x)}{v(x)}\right) = \left(\frac{U}{V}\right)^{(n)}$$

$$= \frac{(-1)^n}{V^{n+1}} \begin{vmatrix} U & V & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U' & V' & V & 0 & 0 & 0 & 0 & 0 & 0 \\ U'' & V'' & V' & V & 0 & 0 & 0 & 0 & 0 \\ U''' & V''' & V'' & V' & V & 0 & 0 & 0 & 0 \\ U^{(4)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 & 0 \\ U^{(5)} & V^{(5)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 \\ & & & & & & V' & V & 0 \\ U^{(n-1)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & V^{(n-5)} & & V' & V \\ U^{(n)} & V^{(n)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & & V'' & V' \end{vmatrix}$$

$$= \frac{1}{(-V)^{n+1}} *$$

$$* \left(\begin{vmatrix} U' & V & 0 & 0 & 0 & 0 \\ U'' & V' & V & 0 & 0 & 0 \\ U''' & V'' & V' & V & 0 & 0 \\ U^{(4)} & V''' & V'' & V' & V & 0 \\ U^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ U^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix} V - U \begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix} \right)$$

$detA_n$ nxn determinants $detB_n$

(12.2)

This formulae cast a long constructivist shadow, including:

3.1 Faa di Bruno's formula

$$\frac{d^n\left(f(v(x))\right)}{n!dx^n} = D^{(n)}\left(f(v(x))\right) = \begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix} = B_n$$

with the substitution

$$V^i \Rightarrow (-1)^i F^{(n-i)} = \left(\frac{d^{n-i} f(y)}{(n-i)!dy^{n-i}} | y=v(x) \right)$$

$$i=0(1)n-1$$

and

$$V^{(i)} = \frac{d^i v(x)}{i!dx^i} \; ; \; i=1(1)n$$

(14.1)

for n = 4

$$\frac{d^4\left(f(v(x))\right)}{4!dx^4} = D^{(4)}\left(f(v(x))\right) = \begin{vmatrix} V' & V & 0 & 0 \\ V'' & V' & V & 0 \\ V''' & V'' & V' & V \\ V^{(4)} & V''' & V'' & V' \end{vmatrix} = B_4$$

$$= V^0V'V'V'V' - 3V^1V'V'V'' + 2V^2V'V''' + V^2V''V'' - V^3V^{(4)}$$

with the substitution

$$V^i \Rightarrow (-1)^i F^{(n-i)} = \left(\frac{d^{n-i} f(y)}{(n-i)!dy^{n-i}} \Big| y=v(x) \right)$$

$$i=0(1)n-1$$

$$= F'V^{(4)} + 2F''V'V'' + F''V''V'' + 3F'''V'V'V'' + F^{(4)}V'V'V'V'$$

Open Problem: Interpret B_n for V = ±1, ±2, … . (14.2,3)

http://mathworld.wolfram.com/FaadiBrunosFormula.html http://en.wikipedia.org/wiki/Fa%C3%A0_di_Bruno's_formula

Johnson, W.P. The Curious History of Faà di Bruno's Formula. American Mathematical Monthly **109** (2002), 217–234. http://www.maa.org/news/monthly217-234.pdf

3.2 Newton's root finding algorithm and more

$$\frac{D^{(n)}\left(\frac{u(x)}{v(x}\right)}{D^{(n)}\left(\frac{1}{v(x}\right)} = \frac{\left(\frac{U}{V}\right)^{(n)}}{\left(\frac{1}{V}\right)^{(n)}} = U - V\frac{\begin{vmatrix} U' & V & 0 & 0 & 0 & 0 \\ U'' & V' & V & 0 & 0 & 0 \\ U''' & V'' & V' & V & 0 & 0 \\ U^{(4)} & V''' & V'' & V' & V & 0 \\ U^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ U^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix}}{\begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix}}$$

$$= U - V\frac{detA_n}{detB_n}$$

(15)

If $v(a) = 0$ and $u(a) \neq 0$ then $\lim_{x \to a} \frac{D^{(n)}\left(\frac{u(x)}{v(x)}\right)}{D^{(n)}\left(\frac{1}{v(x)}\right)} = u(x)$. (16)

Interpret the asignment

$$U := U - V\frac{detA_n}{detB_n} \qquad (17)$$

as body in a recursion loop.

For $u(x) = x$ we obtain the loop body of the extended Newton's root finder of v(x) (18.1)

$$x:=\frac{\left(\frac{x}{v(x)}\right)^{(n)}}{\left(\frac{1}{v(x)}\right)^{(n)}}=x-V\frac{\begin{vmatrix} V' & V & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 \\ V''' & V'' & V' & V & 0 \\ V^{(4)} & V''' & V'' & V' & V \\ V^{(n-1)} & V^{(n-2)} & & & V' \end{vmatrix}}{\begin{vmatrix} V' & V & 0 & 0 & 0 & 0 \\ V'' & V' & V & 0 & 0 & 0 \\ V''' & V'' & V' & V & 0 & 0 \\ V^{(4)} & V''' & V'' & V' & V & 0 \\ V^{(5)} & V^{(4)} & V''' & V'' & V' & V \\ V^{(n)} & V^{(n-1)} & & & & V' \end{vmatrix}} \qquad (18.2)$$

for n=1

$$x:=\frac{\begin{vmatrix} x & V \\ 1 & V' \end{vmatrix}}{\begin{vmatrix} 1 & V \\ 0 & V' \end{vmatrix}}=x-\frac{V}{V'}=x-\frac{v(x)}{v'(x)};\ V^{(i)}=D^{(i)}v(x) \qquad (18.3)$$

n=2

$$x:=\frac{\begin{vmatrix} x & V & 0 \\ 1 & V' & V \\ 0 & V'' & V' \end{vmatrix}}{\begin{vmatrix} 1 & V & 0 \\ 0 & V' & V \\ 0 & V'' & V' \end{vmatrix}}=x-\frac{VV'}{\begin{vmatrix} V' & V \\ V'' & V' \end{vmatrix}}=x-\frac{1}{\frac{V'}{V}-\frac{V''}{V'}} \qquad (18.4)$$

n=3

$$x := \frac{\begin{vmatrix} x & V & 0 & 0 \\ 1 & V' & V & 0 \\ 0 & V'' & V' & V \\ 0 & V''' & V'' & V' \end{vmatrix}}{\begin{vmatrix} 1 & V & 0 & 0 \\ 0 & V' & V & 0 \\ 0 & V'' & V' & V \\ 0 & V''' & V'' & V' \end{vmatrix}} = x - \frac{V\begin{vmatrix} V' & V \\ V'' & V' \end{vmatrix}}{\begin{vmatrix} V' & V & 0 \\ V'' & V' & V \\ V''' & V'' & V' \end{vmatrix}} \qquad (18.5)$$

Convergence!

3.3 How about the special cases u(x) = v'(x) and v(x) = u'(x) ?

3.3.1 u(x) = v'(x)

$$\frac{d^n\left(\frac{v'(x)}{v(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{v'(x)}{v(x)}\right) = \left(\frac{V'}{V}\right)^{(n)}$$

$$= \frac{(-1)^n}{V^{n+1}}\begin{vmatrix} 1 & ' & V & V & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & '' & V & V' & V & 0 & 0 & 0 & 0 & 0 & 0 \\ 3V''' & & & V'' & V' & V & 0 & 0 & 0 & 0 & 0 \\ 4V^{(4)} & & & V''' & V'' & V' & V & 0 & 0 & 0 & 0 \\ 5V^{(5)} & & & V^{(4)} & V''' & V'' & V' & V & 0 & 0 & 0 \\ 6V^{(6)} & & & V^{(5)} & V^{(4)} & V''' & V'' & V' & V & 0 & 0 \\ & & & & & & & & V' & V & 0 \\ nV^{(n)} & & & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & V^{(n-5)} & & V' & V \\ (n & 1 & V^{(n+1)}) & V^{(n)} & V^{(n-1)} & V^{(n-2)} & V^{(n-3)} & V^{(n-4)} & & V'' & V' \end{vmatrix}$$

Open problem: interpretation of this formula

3.3.1 v(x) =u'(x)

$$\frac{d^n\left(\frac{u(x)}{u'(x)}\right)}{n!dx^n} = D^{(n)}\left(\frac{u(x)}{u'(x)}\right) = \left(\frac{U}{U'}\right)^{(n)} = \frac{(-1)^n}{(U')^{n+1}} *$$

$$* \begin{vmatrix} U & 1\ '\ U & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U' & 2U'' & 1U' & 0 & 0 & 0 & 0 & 0 & 0 \\ U'' & 3U''' & 2U'' & 1U' & 0 & 0 & 0 & 0 & 0 \\ U''' & 4U^{(4)} & 3U''' & 2U'' & 1U' & 0 & 0 & 0 & 0 \\ U^{(4)} & 5U^{(5)} & 4U^{(4)} & 3U''' & 2U'' & 1U' & 0 & 0 & 0 \\ U^{(5)} & 6U^{(6)} & 5U^{(5)} & 4U^{(4)} & 3U''' & 2U'' & 1U' & 0 & 0 \\ & & & & & & 2U'' & 1U' & 0 \\ U^{(n-1)} & nU^{(n)} & (n-1)U^{(n-1)} & & & & & 2U'' & 1U' \\ U^{(n)} & (n+1)U^{(n+1)} & nU^{(n)} & & & & & 3U''' & 2U'' \end{vmatrix}$$

3.4

Open problem: interpretation of this formula.

Taylor series representation of fractional functions

$$f(x) = \frac{u(x)}{v(x)}$$ **at a=0**

$$f(x) = \frac{u(x)}{v(x)} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} F^{(n)}(0) x^n \ ;\ F^{(n)}(0) \,|\text{equ. (12); x=0}\, (21)$$

3.4.1 Special case: v(t) is normalized to v(0) = 1

For the special case v(0) = V(0) = 1 (22.1)
we obtain for f(x) the infinite determinant representations

$$f(x) = -\begin{vmatrix} 0 & x^0 & x^1 & x^2 & x^3 & x^4 & \rightarrow \\ U(0) & 1 & 0 & 0 & 0 & 0 & \\ U'(0) & V'(0) & 1 & 0 & 0 & 0 & \\ U''(0) & V''(0) & V'(0) & 1 & 0 & 0 & \\ U'''(0) & V'''(0) & V''(0) & V'(0) & 1 & 0 & \\ U^{(4)}(0) & V^{(4)}(0) & V'''(0) & V''(0) & V'(0) & 1 & \\ \downarrow & \searrow & & & & & \end{vmatrix}$$

$$= -\begin{vmatrix} 0 & U(0) & U'(0) & U''(0) & U'''(0) & U^{(4)}(0) & \rightarrow \\ x^0 & 1 & V'(0) & V''(0) & V'''(0) & V^{(4)}(0) & \\ x^1 & 0 & 1 & V'(0) & V''(0) & V'''(0) & \\ x^2 & 0 & 0 & 1 & V'(0) & V''(0) & \\ x^3 & 0 & 0 & 0 & 1 & V'(0) & \\ x^4 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & \searrow & & & & & \end{vmatrix} \qquad (22.2)$$

3.4.2

$u(x)=\sin x;\ v(x)=\cos x;\ f(x)=\tan x=\dfrac{\sin x}{\cos x}$ (23.1)

$u(0)=0;\ u'(0)=1;\ u''(0)=1;\ u'''(0)=-1;\ u^{(4)}(0)=0 \rightarrow$ per.4 (23.2)

$v(0)=1;\ v'(0)=0;\ v''(0)=-1;\ v'''(0)=0;\ v^{(4)}(0)=1 \rightarrow$ per.4 (23.3)

$$U^{(i)}(0)=\frac{u^{(i)}(0)}{i!};\quad V^{(i)}(0)=\frac{v^{(i)}(0)}{i!} \qquad (23.4)$$

$$F^{(n)}(0)=\frac{(-1)^n}{1}\underbrace{\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{1!} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & -\frac{1}{2!} & 0 & 1 & 0 & 0 & 0 & 0 & \\ -\frac{1}{3!} & 0 & -\frac{1}{2!} & 0 & 1 & 0 & 0 & 0 & \\ 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & 1 & 0 & 0 & \\ \frac{1}{5!} & 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & 1 & 0 & \\ 0 & -\frac{1}{6!} & 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & 1 & \\ -\frac{1}{7!} & 0 & -\frac{1}{6!} & 0 & \frac{1}{4!} & 0 & -\frac{1}{2!} & 0 & \\ & & & & \vdots & & & & \ddots \end{vmatrix}}_{(n+1)\times(n+1)\text{ determinant}} \tag{23.5}$$

From equ. (23.5) follows

$$F^{(0)}(0)=F^{(2)}(0)=F^{(4)}(0)=F^{(6)}(0)=\cdots=F^{(2k)}(0)=0 \tag{23.6}$$

$$F^{(1)}(0)=+\left|\frac{1}{1!}\right|=1 \tag{23.8}$$

$$F^{(3)}(0)=-\begin{vmatrix} \frac{1}{1!} & 1 \\ -\frac{1}{3!} & -\frac{1}{2!} \end{vmatrix}=\frac{1}{3} \tag{23.9}$$

$$F^{(5)}(0)=+\begin{vmatrix} \frac{1}{1!} & 1 & 0 \\ -\frac{1}{3!} & -\frac{1}{2!} & 1 \\ \frac{1}{5!} & \frac{1}{4!} & -\frac{1}{2!} \end{vmatrix}=\frac{2}{15} \tag{23.10}$$

$$F^{(7)}(0) = -\begin{vmatrix} \frac{1}{1!} & 1 & 0 & 0 \\ -\frac{1}{3!} & -\frac{1}{2!} & 1 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & -\frac{1}{2!} & 1 \\ -\frac{1}{7!} & -\frac{1}{6!} & \frac{1}{4!} & -\frac{1}{2!} \end{vmatrix} = \frac{17}{315} \qquad (23.11)$$

So we obtain finally

$$\tan x = \left|\frac{1}{1!}\right| x + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^5 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} x^7 + \ldots$$

$$= \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\searrow & & & & & & \end{vmatrix}$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$

(24)

3.4.2

$u(x)=1,\ v(x)=\cos x;\ f(x)=\dfrac{1}{\cos x}$

$$\frac{1}{\cos x} = 1 + \begin{vmatrix}\frac{1}{2!}\end{vmatrix}x^2 + \begin{vmatrix}\frac{1}{2!} & \frac{1}{0!}\\ \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}x^4 + \begin{vmatrix}\frac{1}{2!} & \frac{1}{0!} & 0\\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!}\\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}x^6 + \begin{vmatrix}\frac{1}{2!} & \frac{1}{0!} & 0 & 0\\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0\\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!}\\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!}\end{vmatrix}x^8 + \ldots$$

$$= \begin{vmatrix} +x^0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow\\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\searrow & & & & & & \end{vmatrix}$$

$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{1385}{40320}x^8 + \ldots$$

(25)

Extensions

$$\frac{1}{\cos^2 x} = 1 + \left|\frac{2}{2!}\right| x^2 + \begin{vmatrix} \frac{2}{2!} & \frac{1}{0!} \\ \frac{8}{4!} & \frac{2}{2!} \end{vmatrix} x^4 + \begin{vmatrix} \frac{2}{2!} & \frac{1}{0!} & 0 \\ \frac{8}{4!} & \frac{2}{2!} & \frac{1}{0!} \\ \frac{32}{6!} & \frac{8}{4!} & \frac{2}{2!} \end{vmatrix} x^6 + \begin{vmatrix} \frac{2}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{8}{4!} & \frac{2}{2!} & \frac{1}{0!} & 0 \\ \frac{32}{6!} & \frac{8}{4!} & \frac{2}{2!} & \frac{1}{0!} \\ \frac{128}{8!} & \frac{32}{6!} & \frac{8}{4!} & \frac{2}{2!} \end{vmatrix} x^8 + \ldots$$

$$= 1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \frac{62}{315}x^8 + \ldots$$

(26)

$$\frac{1}{\cos^3 x} = 1 + \left|A_2\right| x^2 + \begin{vmatrix} A_2 & 1 \\ A_4 & A_2 \end{vmatrix} x^4 + \begin{vmatrix} A_2 & 1 & 0 \\ A_4 & A_2 & 1 \\ A_6 & A_4 & A_2 \end{vmatrix} x^6 + \begin{vmatrix} A_2 & 1 & 0 & 0 \\ A_4 & A_4 & 1 & 0 \\ A_6 & A_2 & A_2 & 1 \\ A_8 & A_6 & A_4 & A_2 \end{vmatrix} x^8 + \ldots$$

$$A_{2k} = \frac{\frac{3}{4}(3^{2k-1}+1)}{(2k)!} \;;\; k=1,2,3,4,....$$

$$A_2 = \frac{3}{2!} \;;\; A_4 = \frac{21}{4!} \;;\; A_6 = \frac{183}{6!} \;;\; A_8 = \frac{1641}{8!} \;;\; ....$$

3.4.3 (27)

$$u(x)=1,\ v(x)=\frac{\sin x}{x}\ ;\ f(x)=\frac{1}{\frac{\sin x}{x}}$$

$$\frac{x}{\sin x} = 1 + \begin{vmatrix}\frac{1}{3!}\end{vmatrix}x^2 + \begin{vmatrix}\frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^4 + \begin{vmatrix}\frac{1}{3!} & \frac{1}{1!} & 0\\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^6 + \begin{vmatrix}\frac{1}{3!} & \frac{1}{1!} & 0 & 0\\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0\\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!}\\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!}\end{vmatrix}x^8 + \ldots$$

$$= \begin{vmatrix} +x^0 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow\\ -x^2 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^4 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^6 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^8 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{10} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\ \searrow & & & & & & \end{vmatrix}$$

$$= 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \frac{31}{15120}x^6 + \frac{127}{604800}x^8 + \ldots$$

(28)

3.4.4 Gudermannian

Gudermannian function

$gd(x) = \tan^{-1}(\sinh(x))$

$$= x - \left|\frac{1}{2!}\right|\frac{x^3}{3} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\frac{x^5}{5} - \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\frac{x^7}{7} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\frac{x^9}{9} \mp \ldots$$

$$= x - \frac{1}{6}x^3 + \frac{1}{24}x^5 - \frac{61}{5040}x^7 + \frac{277}{72576}x^9 \mp \ldots$$

Inverse Gudermannian function

$$gd^{-1}(x) = \ln(\tan x + \frac{1}{\cos x}) = \frac{gd(\sqrt{-1}\,x)}{\sqrt{-1}}$$

$$= x + \left|\frac{1}{2!}\right|\frac{x^3}{3} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\frac{x^5}{5} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\frac{x^7}{7} + \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}\frac{x^9}{9} + \ldots$$

$$= x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \frac{61}{5040}x^7 + \frac{277}{72576}x^9 + \ldots$$

(29.1.2)

$$\left.\begin{matrix} gd(x) \\ gd^{-1}(x) \end{matrix}\right\} = \begin{vmatrix} \frac{x^1}{1} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ \pm\frac{x^3}{3} & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ \frac{x^5}{5} & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ \pm\frac{x^7}{7} & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ \frac{x^9}{9} & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \pm\frac{x^{11}}{11} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\searrow & & & & & & \end{vmatrix} \qquad (29.3)$$

3.4.5

$$e^x = 1+\left|\frac{1}{1!}\right|x + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}x^2 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}x^3 + \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix}x^4 + \ldots$$

$$= 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\ldots$$

.

$$= \begin{vmatrix} +x^0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ +x^2 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ -x^3 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ +x^4 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & \searrow \end{vmatrix} \qquad (30.1,2)$$

3.4.6

$$\frac{x}{\ln(1+x)} = 1 + \left|\frac{1}{2}\right| x + \begin{vmatrix} \frac{1}{2} & \frac{1}{1} \\ \frac{1}{3} & \frac{1}{2} \end{vmatrix} x^2 + \begin{vmatrix} \frac{1}{2} & \frac{1}{1} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{vmatrix} x^3 + \begin{vmatrix} \frac{1}{2} & \frac{1}{1} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{1} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{1}{1} \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \end{vmatrix} x^4 + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{24}x^3 - \frac{19}{720}x^4 + \frac{3}{160}x^5 - \frac{863}{60480}x^6 + \frac{1375}{120960}x^7 + \frac{275}{24192}x^8 - \frac{33953}{3628800}x^9 + \frac{8183}{1036800}x^{10} - \frac{3250433}{479001600}x^{11} + \frac{4671}{788480}x^{12} - \ldots \qquad (31)$$

3.4.7

$$u(x) = \sum_{i=0}^{\infty} a_i x^i;\ \mathrm{v}(x) = \sum_{i=0}^{\infty} b_i x^i \qquad (32.1)$$

$$U^{(i)}(0) = a_i;\ V^{(i)}(0) = b_i \qquad (32.2)$$

$$F^{(n)}(0)=\frac{(-1)^n}{b_0^{n+1}}\begin{vmatrix} a_0 & b_0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & b_1 & b_0 & 0 & \cdots & 0 & 0 \\ a_2 & b_2 & b_1 & b_0 & & 0 & 0 \\ a_3 & b_3 & b_2 & b_1 & \ddots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ a_{n-1} & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_1 & b_0 \\ a_n & b_n & b_{n-1} & b_{n-2} & \cdots & b_2 & b_1 \end{vmatrix} \qquad (32.3)$$

$$F(0)=\frac{a_0}{b_0} \qquad (32.4)$$

$$F'(0)=\frac{-1}{b_0^2}\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \qquad (32.5)$$

$$F''(0)=\frac{1}{b_0^3}\begin{vmatrix} a_0 & b_0 & 0 \\ a_1 & b_1 & b_0 \\ a_2 & b_2 & b_1 \end{vmatrix} \qquad (32.6)$$

$$\frac{\sum_{i=0}^{\infty} a_i x^i}{\sum_{i=0}^{\infty} b_i x^i}=\sum_{n=0}^{\infty} F^{(n)}(0)x^n \qquad (32.7)$$

$$=-\begin{vmatrix} 0 & a_0 & a_1 & a_2 & a_3 & \rightarrow \\ x^0 & 1 & b_1 & b_2 & b_3 & \\ x^1 & 0 & 1 & b_1 & b_2 & \\ x^2 & 0 & 0 & 1 & b_1 & \\ x^3 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix} \quad \text{with } b_0 = 1 \qquad (32.8)$$

Inversion for $a_0=1$; $(\forall i)\ a_n=0$, $n=1(1)\infty$

$$b_0=\frac{1}{F^{(0)}(0)}$$

$$b_n=\frac{(-1)^n}{(F^{(0)}(0))^{n+1}}\begin{vmatrix} 1 & F^{(0)}(0) & 0 & \cdots & 0 \\ 0 & F^{(1)}(0) & F^{(0)}(0) & \cdots & 0 \\ 0 & F^{(2)}(0) & F^{(1)}(0) & \cdots & 0 \\ 0 & F^{(3)}(0) & F^{(2)}(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & F^{(n-1)}(0) & F^{(n-2)}(0) & \cdots & F^{(0)}(0) \\ 0 & F^{(n)}(0) & F^{(n-1)}(0) & \cdots & F^{(1)}(0) \end{vmatrix} \quad (33)$$

3.4.8

$$u(x)=(1+x)^k-(1-x)^k;\ \mathrm{v}(x)=(1+x)^k+(1-x)^k \quad (34.1)$$

With

$$U^{(i)}(0)=\frac{1}{i!}u^{(i)}(0);\ V^{(i)}(0)=\frac{1}{i!}v^{(i)}(0) \quad (34.2)$$

we have

$$U^{(2l)}(0)=0,\ U^{(2l+1)}(0)=2\binom{k}{2l+1};\ l=0,1,2,\cdots \quad (34.3)$$

$$V^{(2l)}(0)=2\binom{k}{2l},\ \mathrm{V}^{(2l+1)}(0)=0;\ l=0,1,2,\cdots \quad (34.4)$$

and

$$F^{(n)}(0 = \underbrace{\begin{vmatrix} 0 & \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & 0 & \\ \binom{k}{1} & 0 & \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & \\ 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & 0 & 0 & 0 & \\ \binom{k}{3} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & 0 & 0 & \\ 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & 0 & \\ \binom{k}{5} & 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & 0 & \\ 0 & \binom{k}{6} & 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \binom{k}{0} & \\ \binom{k}{7} & 0 & \binom{k}{6} & 0 & \binom{k}{4} & 0 & \binom{k}{2} & 0 & \\ & & & & \vdots & & & & \ddots \end{vmatrix}}_{(n+1)\times(n+1)\text{ determinant}}$$

(34.5)

With

$$F^{(2l)}(0) = 0 \tag{34.6}$$

$$\begin{vmatrix} \binom{k}{1} & \binom{k}{0} & 0 & & 0 \\ \binom{k}{3} & \binom{k}{2} & \binom{k}{0} & & 0 \\ \binom{k}{5} & \binom{k}{4} & \binom{k}{2} & & \end{vmatrix}$$

$$(l+1)\times(l+1)\ \text{determinant} \qquad (34.7)$$

we obtain finally

$$f(x)=\frac{(1+x)^k-(1-x)^k}{(1+x)^k+(1-x)^k}=\binom{k}{1}x-\begin{vmatrix}\binom{k}{1} & \binom{k}{0}\\ \binom{k}{3} & \binom{k}{2}\end{vmatrix}x^3+\begin{vmatrix}\binom{k}{1} & \binom{k}{0} & 0\\ \binom{k}{3} & \binom{k}{2} & \binom{k}{0}\\ \binom{k}{5} & \binom{k}{4} & \binom{k}{2}\end{vmatrix}x^5$$

$$-\begin{vmatrix}\binom{k}{1} & \binom{k}{0} & 0 & 0\\ \binom{k}{3} & \binom{k}{2} & \binom{k}{0} & 0\\ \binom{k}{5} & \binom{k}{4} & \binom{k}{2} & \binom{k}{0}\\ \binom{k}{7} & \binom{k}{6} & \binom{k}{4} & \binom{k}{2}\end{vmatrix}x^7\pm\cdots \qquad (35.1)$$

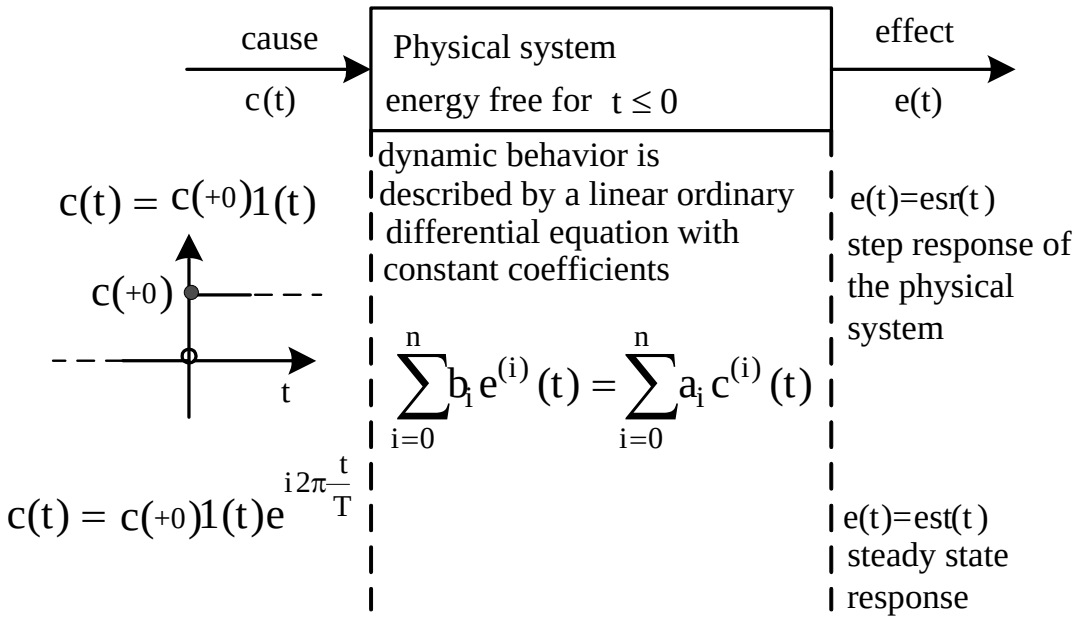
$$= -\begin{vmatrix} 0 & \binom{k}{1} & \binom{k}{3} & \binom{k}{5} & \binom{k}{7} & \rightarrow \\ x^1 & \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \binom{k}{6} & \\ x^3 & 0 & \binom{k}{0} & \binom{k}{2} & \binom{k}{4} & \\ x^5 & 0 & 0 & \binom{k}{0} & \binom{k}{2} & \\ x^7 & 0 & 0 & 0 & \binom{k}{0} & \\ \downarrow & & & & & \searrow \end{vmatrix} \tag{35.2}$$

Note

$$\binom{k}{i} = \begin{cases} \dfrac{k!}{i!(k-i)!} & k \geq i \\ 0 & k < i \end{cases} \quad \text{iff} \quad ;\ k,i \in \mathbb{N} \tag{36}$$

4. Solution of linear differential equations, esp. applications in electrical engineering

Given



(37)

* Step response esr(t) and
* steady state response est(t) on a complex exponential (rotation operator); frequency response

are the most important dynamic input-output relations for the evaluation of such a system.

Find

Taylor series representation of esr(t) via Laplace transform

Solution

$$\frac{esr(t)}{c(+0)} = \sum_{k=0}^{\infty} \frac{esr^{(k)}(+0)}{c(+0)} \frac{t^k}{k!} \; ; t \geq 0$$

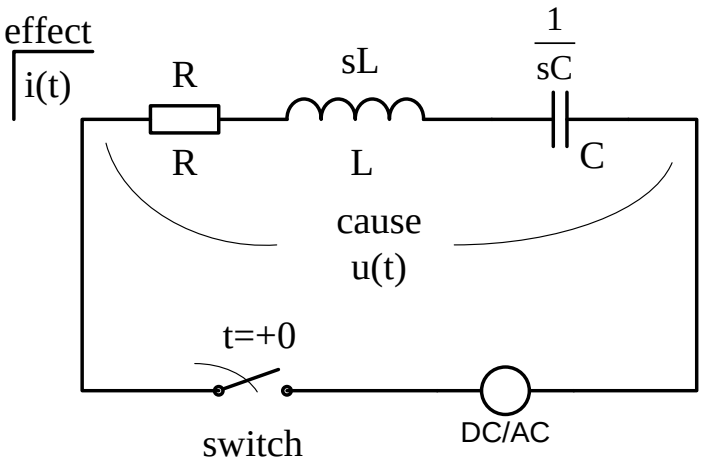
$$\frac{esr^{(k)}(+0)}{c(+0)} = \frac{(-1)^k}{b_n^{k+1}} \begin{vmatrix} a_n & b_n & 0 & 0 & 0 & 0 \\ a_{n-1} & b_{n-1} & 0 & 0 & 0 & \\ a_{n-2} & b_{n-1} & b_{n-1} & 0 & 0 & \\ a_{n-3} & b_{n-1} & b_{n-1} & b_{n-1} & 0 & \\ a_{n-4} & b_{n-1} & b_{n-1} & b_{n-1} & b_{n-1} & \\ & & & & & \\ a_{n-k} & b_{n-1} & b_{n-1} & & & b_{n-1} \end{vmatrix}$$

(k+1)x(k+1) determinant

(38.1,2)

$$\left.\frac{\mathrm{esr}(t)}{\mathrm{c}(+0)}\right|_{b_n=1} = -\begin{vmatrix} 0 & a_n & a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \rightarrow \\ \frac{t^0}{0!} & 1 & b_{n-1} & b_{n-2} & b_{n-3} & b_{n-4} & \\ \frac{t^1}{1!} & 0 & 1 & b_{n-1} & b_{n-2} & b_{n-3} & \\ \frac{t^2}{2!} & 0 & 0 & 1 & b_{n-1} & b_{n2} & \\ \frac{t^3}{3!} & 0 & 0 & 0 & 1 & b_{n-1} & \\ \frac{t^4}{4!} & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

(38.3)



Example: electrical oscillatory circuit

energy free for t≤0

$\rightarrow i_L(+0) = i(+0) = 0$

$u_C(+0) = 0$

$$i''(t) + \frac{R}{L} i'(t) + \frac{1}{LC} i(t) = \frac{1}{L} u'(t) \quad ; \ \mathrm{L,C} \neq 0$$

n=2; $a_2 = 0$; $a_1 = \frac{1}{L}$; $a_0 = 0$

$$b_2 = 1;\ b_1 = \frac{R}{L};\ b_0 = \frac{1}{LC} \qquad (39)$$

“Screenplay” solution

$$isr(t) = u(t)\left(\frac{1}{L}t - \frac{R}{L^2}\frac{t^2}{2!} + \frac{1}{L}\begin{vmatrix} \frac{R}{L} & 1 \\ \frac{1}{LC} & \frac{R}{L} \end{vmatrix}\frac{t^3}{3!} - \frac{1}{L}\begin{vmatrix} \frac{R}{L} & 1 & 0 \\ \frac{1}{LC} & \frac{R}{L} & 1 \\ 0 & \frac{1}{LC} & \frac{R}{L} \end{vmatrix}\frac{t^4}{4!} \pm \cdots \right) \qquad (40)$$

Normalized solution

$$isr(t) = \frac{u(t)}{R} *$$

$$* \left(\frac{Rt}{L} - \left(\frac{Rt}{L}\right)^2 \frac{1}{2!} + \begin{vmatrix} 1 & 1 \\ \frac{L}{R^2C} & 1 \end{vmatrix}\left(\frac{Rt}{L}\right)^3 \frac{1}{3!} - \begin{vmatrix} 1 & 1 & 0 \\ \frac{L}{R^2C} & 1 & 1 \\ 0 & \frac{L}{R^2C} & 1 \end{vmatrix}\left(\frac{Rt}{L}\right)^4 \frac{1}{4!} \pm \cdots \right) \qquad (41)$$

Normalized branched continued fraction representation

$$isr(t)=\frac{u(t)}{R}\left(\cfrac{x}{1}+\cfrac{\frac{x}{2}}{1-\frac{x}{2}}+\cfrac{\frac{x}{3}\left(1-\frac{\delta}{1}\right)}{\left(\frac{3}{x}-\frac{1}{1}\right)\left(\delta-\frac{1}{1}\right)}+\cfrac{\frac{x}{4}\left(1-\frac{\delta}{1}\right)}{\left(\frac{4}{x}-\frac{1}{1}\right)\left(\delta-\frac{1}{1}\right)}+\cfrac{\frac{x}{5}\left(1-\frac{\delta}{1}\right)}{\left(\frac{5}{x}-\frac{1}{1}\right)\left(\delta-\frac{1}{1}\right)}+\cdots\right)$$

$$x=\frac{R}{L}t,\ \delta=\frac{L}{R^2C}\neq 1$$

(42)

5. A zoo of reciprocals of the factorial determinant patterns

5.1 Connections with Bernoulli and Euler numbers

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n}(2^{2n}-1)} \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix} \quad ; \quad n=1,2,3,4,..$$

determinant of nth order

$$B_{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n}-2} \begin{vmatrix} \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & \rightarrow \\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & \rightarrow \\ \frac{1}{1\ !1} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix} \quad ; \quad n=1,2,3,4,..$$

determinant of nth order

(43.1,2)

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix} \quad ; \quad n=1,2,3,4,...$$

determinant of nth order

(44.1)

$$B_0 = 1;\ B_1 = -\frac{1}{2};\ B_2 = \frac{1}{6};\ B_4 = -\frac{1}{30};\ B_6 = \frac{1}{42};\ B_8 = -\frac{1}{30};\ \ldots$$

$$B_{2k+1} = 0;\ k = 1,2,3,...$$

http://mathworld.wolfram.com/BernoulliNumber.html (43.3)

$$E_2 = -1;\ E_4 = 5;\ E_6 = -61;\ E_8 = 1385;\ E_{10} = -50521;\ ...$$

$$E_{2k+1} = 0;\ k = 0,1,2,3,..$$

http://mathworld.wolfram.com/EulerNumber.html (44.2)

$$\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} \\ \frac{1}{11!} & \frac{1}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix} = \frac{2^{2n}(2^{2n}-1)}{2^{2n}-2} \begin{vmatrix} \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 \\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 \\ \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} \\ \frac{1}{13!} & \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} \end{vmatrix}$$

(45)

5.2 A connection with π

The sequence

$$\begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix}$$

$$g(n) = \frac{\text{determinant nth order}}{\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \end{vmatrix}} \quad ; \quad n=1,2,3,4,\ldots$$

determinant (n+1) order

(46.1)

converges linearly with the difference quotient

$$\lim_{n\to\infty} \frac{g(n)-g(n+1)}{g(n+1)-g(n+2)} = 9$$

to

$$\lim_{n\to\infty} g(n) = \frac{\pi}{2} \; .$$

(46.2,3)

5.3 A connection with e

$$e = \begin{vmatrix} +1 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ -1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ +1 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ -1 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ +1 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

$$\frac{1}{e} = \begin{vmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \rightarrow \\ 1 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \\ 1 & 0 & \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \\ 1 & 0 & 0 & \frac{1}{0!} & \frac{1}{1!} & \\ 1 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & \searrow \end{vmatrix}$$

(47.1,3)

5.4 Connections with other mathematical constants

5.5 Connections with prines

5.6 Miscellaneous

$$\frac{1}{\frac{1}{n!}}\begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} & 0 \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-2)!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & \frac{1}{0!} \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} = 1 \quad ; n = 1,2,3,..$$

nxn determinant

(48)

$$\frac{1}{\frac{1}{n!}}\begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} = B_n \quad ; n = 2,3,4,$$

nxn determinant

(49)

$$\frac{1}{\frac{1}{n!}}\begin{vmatrix} \frac{1}{3!} & \frac{1}{2!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 & 0 & 0 \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 & 0 \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 0 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \\ \frac{1}{(n+2)!} & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} \end{vmatrix} = ???(n) \quad ; n = 1,2.3,..$$

nxn determinant

6. Sum of powers of natural numbers

6.1 Sum

$$s(n,m) = \sum_{i=1}^{n} i^m \; ; \quad s(n,0) = n \; ; \; s(n,1) = \frac{1}{2}n(n+1) \; ;$$

$$s(n,2) = \frac{1}{6}n(n+1)(2n+1) \; ;$$

$$s(n,3) = \left(\frac{1}{2}n(n+1)\right)^2 = (s(n,1))^2$$

$$\frac{s(n,m)}{s(n,0)} =$$

$$\frac{1}{(m+1)!}\begin{vmatrix}
(-n)^0 & \binom{1}{0} & 0 & 0 & 0 & 0 & 0 \\
(-n)^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 & 0 & 0 \\
(-n)^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 & 0 & 0 \\
(-n)^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & & 0 & 0 \\
 & & & & & & 0 \\
(-n)^{m-1} & \binom{m}{0} & \binom{m}{1} & \binom{m}{2} & & \binom{m}{m-2} & \binom{m}{m-1} \\
(-n)^m & \binom{m+1}{0} & \binom{m+1}{1} & \binom{m+1}{2} & & \binom{m+1}{m-2} & \binom{m+1}{m-1}
\end{vmatrix}$$

(m+1)x(m+1) determinant

6.1.1 Sum of even powers

$$\frac{f(n,2k)}{f(n,2)} = \frac{\sum_{i=1}^{n} i^{2k}}{\sum_{i=1}^{n} i^{2}} = \frac{(-1)^{k+1}}{\prod_{i=2}^{k}(2i+1)} *$$

$$* \begin{vmatrix}
p^0 & \binom{1}{0}+\binom{2}{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
p^1 & \binom{3}{0} & \binom{2}{1}+\binom{3}{2} & 0 & 0 & 0 & 0 & 0 \\
p^2 & 0 & \binom{3}{0}+\binom{4}{1} & \binom{3}{2}+\binom{4}{3} & 0 & 0 & 0 & 0 \\
p^3 & 0 & \binom{5}{0} & \binom{4}{1}+\binom{5}{2} & \binom{4}{3}+\binom{5}{4} & 0 & 0 & 0 \\
p^4 & 0 & 0 & \binom{5}{0}+\binom{6}{1} & \binom{5}{2}+\binom{6}{3} & \binom{5}{4}+\binom{6}{5} & 0 & 0 \\
p^5 & 0 & 0 & \binom{7}{0} & \binom{6}{1}+\binom{7}{2} & \binom{6}{3}+\binom{7}{4} & \binom{6}{5}+\binom{7}{6} & 0 \\
p^6 & 0 & 0 & 0 & \binom{7}{0}+\binom{8}{1} & \binom{7}{2}+\binom{8}{3} & \binom{7}{4}+\binom{8}{5} & \binom{7}{6}+\binom{8}{7} \\
p^7 & 0 & 0 & 0 & \binom{9}{0} & \binom{8}{1}+\binom{9}{2} & \binom{8}{3}+\binom{9}{4} & \binom{8}{5}+\binom{9}{6}
\end{vmatrix}$$

kxk determinant

$p = n(n+1)$

(52)

6.1.2 Sum of odd powers

$$\frac{f(n,2k+1)}{f(n,3)} = \frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^{3}} =$$

$$\frac{2(-1)^{k+1}}{(k+1)!} \begin{vmatrix}
p^0 & \binom{2}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^1 & \binom{3}{0} & \binom{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^2 & 0 & \binom{4}{1} & \binom{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p^3 & 0 & \binom{5}{0} & \binom{5}{2} & \binom{5}{4} & 0 & 0 & 0 & 0 & 0 \\
p^4 & 0 & 0 & \binom{6}{1} & \binom{6}{3} & \binom{6}{5} & 0 & 0 & 0 & 0 \\
p^5 & 0 & 0 & \binom{7}{0} & \binom{7}{2} & \binom{7}{4} & \binom{7}{6} & 0 & 0 & 0 \\
p^6 & 0 & 0 & 0 & \binom{8}{1} & \binom{8}{3} & \binom{8}{5} & \binom{8}{7} & 0 & 0 \\
p^7 & 0 & 0 & 0 & \binom{9}{0} & \binom{9}{2} & \binom{9}{4} & \binom{9}{6} & \binom{9}{8} & 0 \\
p^8 & 0 & 0 & 0 & 0 & \binom{10}{1} & \binom{10}{3} & \binom{10}{5} & \binom{10}{7} & \binom{10}{9} \\
p^9 & 0 & 0 & 0 & 0 & \binom{11}{0} & \binom{11}{2} & \binom{11}{4} & \binom{11}{6} & \binom{11}{8}
\end{vmatrix}$$

kxk determinant

$p = n(n+1)$

(53)

6.2 ±oscillating sum

$$os(n,m) = \sum_{i=1}^{n} (-1)^{n-i} i^m \;\; ; \quad os(n,0) = \begin{cases} 0 \\ 1 \end{cases} \text{ for n is } \begin{cases} \text{even} \\ \text{odd} \end{cases} ;$$

$$os(n,1) = \text{int}\left(\frac{n+1}{2}\right) ; \qquad (54)$$

$$os(n,2) = \frac{1}{2} n(n+1) = s(n,1)$$

6.2.1 ±oscillating sum of even powers

$$\frac{os(n,2k)}{os(n,2)} = \frac{\sum_{i=1}^{n}(-1)^{n-i} i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i} i^{2}}$$

$$= (-1)^{k+1} \begin{vmatrix}
p^0 & \binom{1}{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^1 & \binom{2}{0} & \binom{2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^2 & 0 & \binom{3}{1} & \binom{3}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p^3 & 0 & \binom{4}{0} & \binom{4}{2} & \binom{4}{4} & 0 & 0 & 0 & 0 & 0 \\
p^4 & 0 & 0 & \binom{5}{1} & \binom{5}{3} & \binom{5}{5} & 0 & 0 & 0 & 0 \\
p^5 & 0 & 0 & \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & \binom{6}{6} & 0 & 0 & 0 \\
p^6 & 0 & 0 & 0 & \binom{7}{1} & \binom{7}{3} & \binom{7}{5} & \binom{7}{7} & 0 & 0 \\
p^7 & 0 & 0 & 0 & \binom{8}{0} & \binom{8}{2} & \binom{8}{4} & \binom{8}{6} & \binom{8}{8} & 0 \\
p^8 & 0 & 0 & 0 & 0 & \binom{9}{1} & \binom{9}{3} & \binom{9}{5} & \binom{9}{7} & \binom{9}{9} \\
p^9 & 0 & 0 & 0 & 0 & \binom{10}{0} & \binom{10}{2} & \binom{10}{4} & \binom{10}{6} & \binom{10}{8}
\end{vmatrix}$$

kxk determinant

$p = n(n+1)$

(55)

6.2.2 ±oscillating sum of odd powers

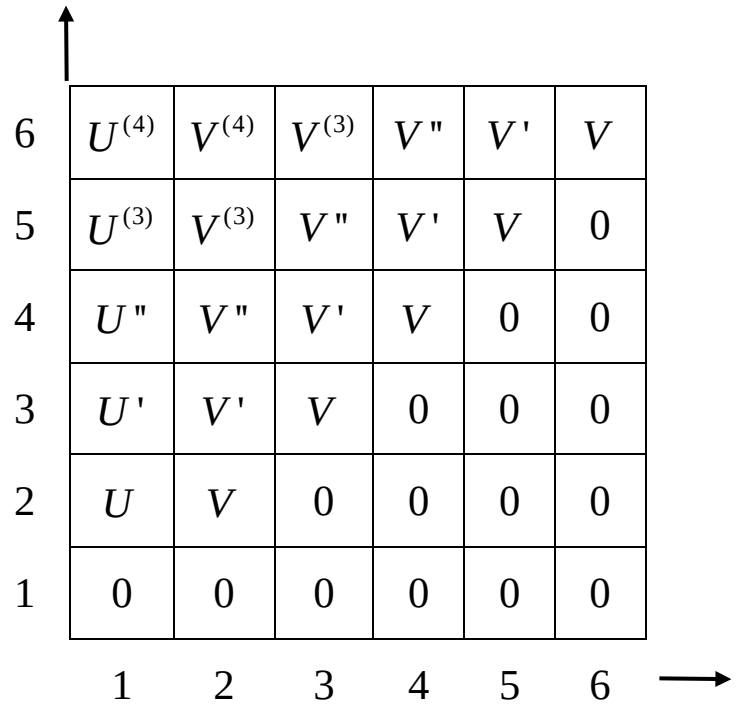
Open problem

6.3 Other oscillation patterns sum

7. Extended quotient rule in an algorithmic-oriented representation (array + navigation rule + local operator)

Cartesian mxm-array (2D pattern processing algorithms), here m=6

Initial setting:



Algorithm for the simultaneous sequential calculation of $F^{(0)}$, $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, $F^{(4)}$, …

b_0:=U

$F^{(0)} := \frac{b_0}{V}$

j=2(1)m-1 navigation

k=j+1(1)m rule

c[1,k]:=c[j,j]*c[1,k]-c[1,j]*c[j,k] local operator

↑ ↑

new old content of cell[1,k]

b_{j-1}:=c[1,j+1]

$F^{(j-1)} := \frac{b_{j-1}}{V^{j}}$

Starting from j=2 by step 1
the old column[1] above j is overwritten with
(content of diagonal cell[j,j] times old column[1] above j minus content of cell[1,j] times column[j] above j).

A very important algorithm for many applications! What happens when we repeat this procedure?

Final setting of the array

6	$b_4 = V\left(V\left(V\left(VU^{(4)} - b_0V^{(4)}\right) - b_1V^{(3)}\right) - b_2V''\right) - b_3V'$ $V\left(V\left(VU^{(4)} - b_0V^{(4)}\right) - b_1V^{(3)}\right) - b_2V''$ $V\left(VU^{(4)} - b_0V^{(4)}\right) - b_1V^{(3)}$ $VU^{(4)} - b_0V^{(4)}$	$V^{(4)}$	$V^{(3)}$	V''	V'	V
5	$b_3 = V\left(V\left(VU^{(3)} - b_0V^{(3)}\right) - b_1V''\right) - b_2V'$ $V\left(VU^{(3)} - b_0V^{(3)}\right) - b_1V''$	$V^{(3)}$	V''	V'	V	0

	$VU^{(3)}-b_0V^{(3)}$					
4	$b_2=V\left(VU''-b_0V''\right)-b_1V'$ $VU''-b_0V''$	V''	V'	V	0	0
3	$b_1=VU'-b_0V'$	V'	V	0	0	0
2	$b_0=U$	V	0	0	0	0
1	0	0	0	0	0	0
	1	2	3	4	5	6

$$b_4=\left(\left(\left(\left(U^{(4)}\right)V-b_0V^{(4)}\right)V-b_1V^{(3)}\right)V-b_2V''\right)V-b_3V'$$
$$=U^{(4)}VVVV-U^{(3)}VVVV'-U''VVVV''+U''VVV'V'-U'VVVV^{(3)}$$
$$+U'VVV'V''+U'VVV'V''-U'VV'V'V'-UVVVV^{(4)}+UVVV'V^{(3)}$$
$$+UVVV'V^{(3)}+UVVV''V''-UVV'V'V''-UVV'V'V''-UVV'V'V''$$
$$+UV'V'V'V'$$
$$b_3=\left(\left(\left(U^{(3)}\right)V-b_0V^{(3)}\right)V-b_1V''\right)V-b_2V'$$
$$=U^{(3)}VVV-U''VVV'-U'VVV''+U'VV'V'-UVVV^{(3)}+UVV'V''$$
$$+UVV'V''-UV'V'V'$$
$$b_2=\left(\left(U''\right)V-b_0V''\right)V-b_1V'$$
$$=U''VV-U'VV'-UVV''+UV'V'$$
$$b_1=\left(U'\right)V-b_0V'$$
$$=U'V-UV'$$
$$b_0=U$$