Unit 19 19

Computational geometry. A synoptic shortened determinant representation of area, perimeter, centroid, momentum of inertia, etc. of a 2D polygon given by the Cartesian coordinates of its vertices.

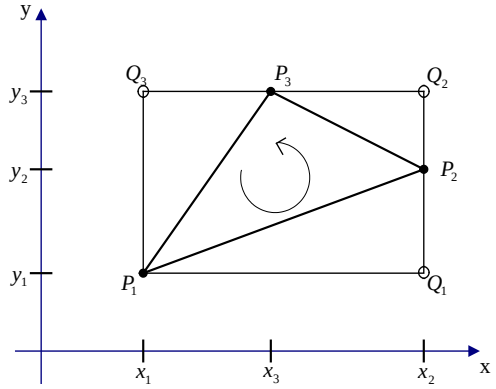
Lecture given at the Sichuan University, Chengdu

Yangtze Center of Mathematics

A problem-oriented 2D/3D notation system for applications in computational geometry, mechanics and other engineering fields: 2D shortened determinant, 3D data cylinder.

1.1 Introduction

We start with a simple geometric problem.



Given a planar triangle $\triangle P_1P_2P_3$ by the coordinates of its vertices $P_i = (x_i, y_i)$; $i = 1(1)3$, the simplest interaction between the points P_1, P_2 and P_3.

Find a formula expression for the area of $\triangle P_1P_2P_3$ in function of the coordinates x_i and y_i taking a simplest and shortest way.

Solution by embedding of $\triangle P_1P_2P_3$ in the minimal rectangle $Q_1Q_2Q_3Q_4$

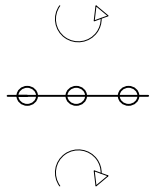
$$\begin{aligned}\text{area}\Delta P_1P_2P_3 &= \text{area}\square P_1Q_1Q_2Q_3 - \text{area}\Delta P_1Q_1P_2 - \text{area}\Delta P_2Q_2P_3 \\ &\quad - \text{area}\Delta P_3Q_3P_1 \\ &= (x_2 - x_1)(y_3 - y_1) - \frac{1}{2}(x_2 - x_1)(y_2 - y_1) \\ &\quad - \frac{1}{2}(x_2 - x_3)(y_3 - y_2) - \frac{1}{2}(x_3 - x_1)(y_3 - y_1) \\ &= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3) \qquad (1)\end{aligned}$$

1D string notation (note the high inner symmetry in this expression!).

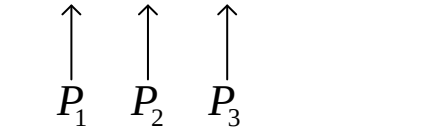
The classical 2D determinant notation of equ. (1) is a much more

convincing reflection of the given geometric object:

$$area_{\triangle}P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff}$$

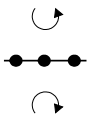


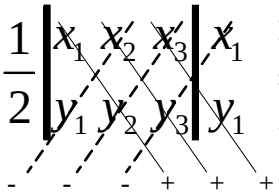
$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_1 & x_2 \\ y_1 & y_2 & y_3 & y_1 & y_2 \end{vmatrix} \begin{matrix} \} \\ \} \end{matrix} \begin{matrix} \text{redundancy} \\ \text{information} \end{matrix} \qquad (2)$$



A very important engineering principle is: cancel unnecessary redundancy in your design . We go this way and cut the redundancy in equ.(2), getting the “shortened 2 by 3 determinant ” notation

$$area_{\triangle}P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff} \qquad (3)$$

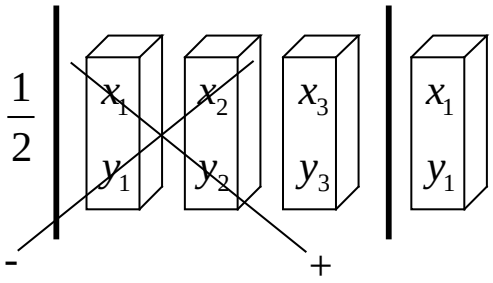




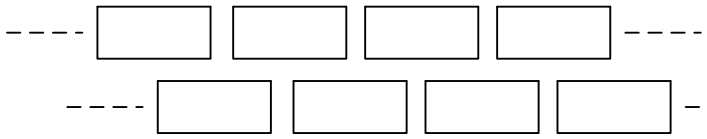
Formula design according to “creative visual resonance”.
Two comments:

1. In my civil engineering inspired imagination, the shortened determinant equ. (3) consists of 3 “bricks” (or tiles) in “grenadier bond”, a special kind of brickwork structure, you can see in walls of houses etc. , each representing a vertex of the $\triangle P_1P_2P_3$:

$$\text{area}_{\triangle}P_1P_2P_3 = \frac{1}{2}\left|\begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix}\right| \begin{matrix} x_1 \\ y_1 \end{matrix} \qquad (4)$$



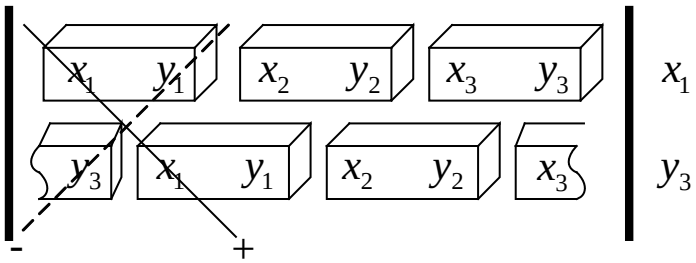
This association implies immediately the civil engineering (not mathematics) born question: how about other kinds of brickwork structures, f.e . ,“stretcher bond”



,

with regard to the $\triangle P_1P_2P_3$, or cut right to the chase: is the brick-idea a productively extensible notation idea?

Let’s have a look to the shortened 2 by 6 determinant:



(5)

The Sarrus evaluation of this determinant gives:

$$
\begin{aligned}
? &= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 \\
&\quad - x_1x_3 - y_1y_3 - x_1x_2 - y_1y_2 - x_2x_3 - y_2y_3 \\
&= \frac{1}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2 + (x_2 - x_3)^2 + (y_2 - y_3)^2 \\
&\qquad + (x_3 - x_1)^2 + (y_3 - y_1)^2) \\
&= \frac{1}{2}(\overline{P_1P_2}^2 + \overline{P_2P_3}^2 + \overline{P_3P_1}^2) \quad \text{of} \quad \triangle P_1P_2P_3\,!
\end{aligned}
\tag{6}
$$

A first hint, maybe we strike oil!

A second: circumcenter coordinates (x_{cc},y_{cc}) of $\triangle P_1P_2P_3$!, equ. (A2)

Later more, and for more about brickwork see:

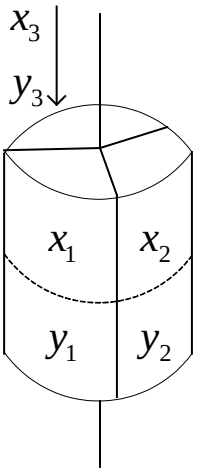
R. Field. Geometric Patterns from Tiles and Brickworks.

Targuin Publications 2004.

Translation into Chinese: ISBN 7 5777 0415 3

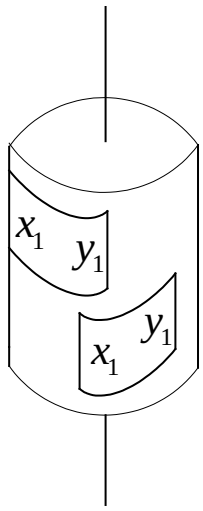
2. Return to the starting point.

Another, mechanical engineering (not mathematics) born interpretation of equ. (3) is the following.
The 2D shortened determinant is a 2D “mapping ” of a 3D grenadier bond data cylinder



with 2 tracks and 3 segments (corresponding to the 2 rows and 3 columns of the shortened determinant), the bricks of the grenadier bond are the cylinder segments!

In the case of stretcher bond



with 2 tracks (corresponding to the 2 rows of the shortened determinant) and 6 segments (corresponding to the 6 stretcher bricks).

To represent the 3D master, we write it in the following:

x_1	x_2	x_3
y_1	y_2	y_3

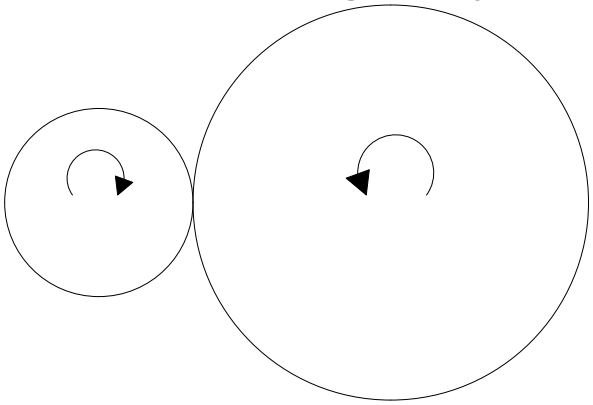
;

x_1	y_1	x_2	y_2	x_3	y_3
y_3	x_1	y_1	x_2	y_2	x_3

(7; 8)

The data cylinder model implies immediately the mechanical engineering born extension:

How about a rotating data cylinder gear?



How about a “Moebius Strip” [1] data cylinder and related (friction, belt, etc.) gears [2]?

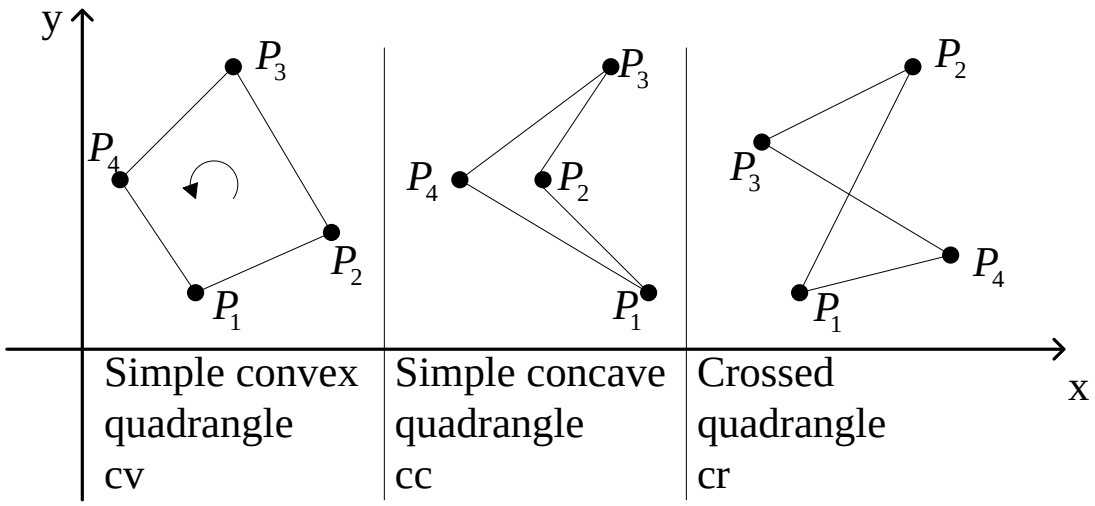
[1] E. W. Weisstein. Moebius Strip.
http://mathworld.wolfram.com/Moebiustrip.html
[2] http://www.answers.com/topic/gear

1.2 A first extension stream: from triangle to quadrangle

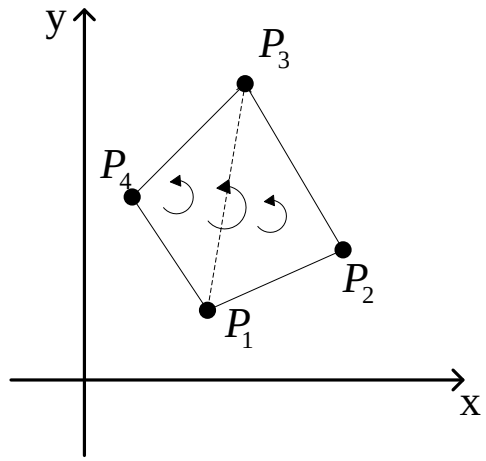
We have in our computational geometry algorithm design toolbox now the “**b**rickwork- oriented shortened **d**eterminant-data **c**ylinder (bdc)” tool. And we must answer again and again on the question: is bdc a mayfly as a tool to strike oil?

1.2.1 Area and sum of the side squares

Let’s go a first extension step from triangle with the result equs (1,2,3) and equs (5,6) to the quadrangle in 2D.

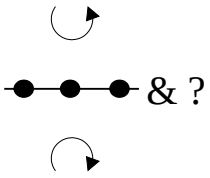


We start with the simple convex quadrangle cv



$$area_{\square} P_1P_2P_3P_4 = area_{\triangle} P_1P_2P_3 + area_{\triangle} P_1P_3P_4$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix} \quad \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff} \quad \text{  & ?} \qquad (9)$$

A convincing extension!

By the way, in classical determinant notation:

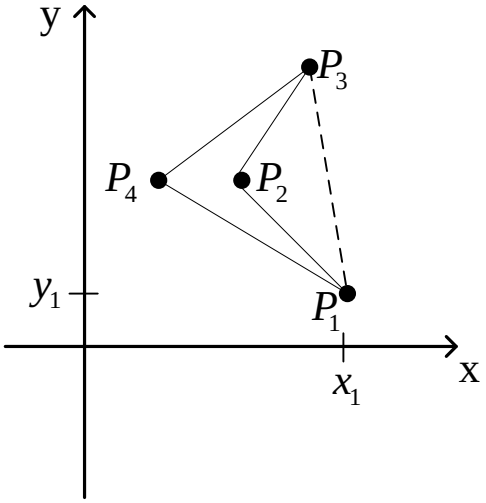
$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} \begin{matrix} \} \text{redundancy} \\ \\ \} \text{information} \end{matrix} \qquad (10)$$

Find similar classical determinant representations for the area of a simple convex 2D polygon (n-gon).

$$\overline{P_1P_2}^2 + \overline{P_2P_3}^2 + \overline{P_3P_4}^2 + \overline{P_4P_1}^2 = 2\begin{vmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 & y_4 \\ y_4 & x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 \end{vmatrix} \qquad (11)$$

Convincing too!
Valid also for cc and cr.

Now the simple concave quadrangle cc.



$$area_{\square}P_1P_2P_3P_4 = area_{\triangle}P_1P_3P_4 - area_{\triangle}P_1P_3P_2$$

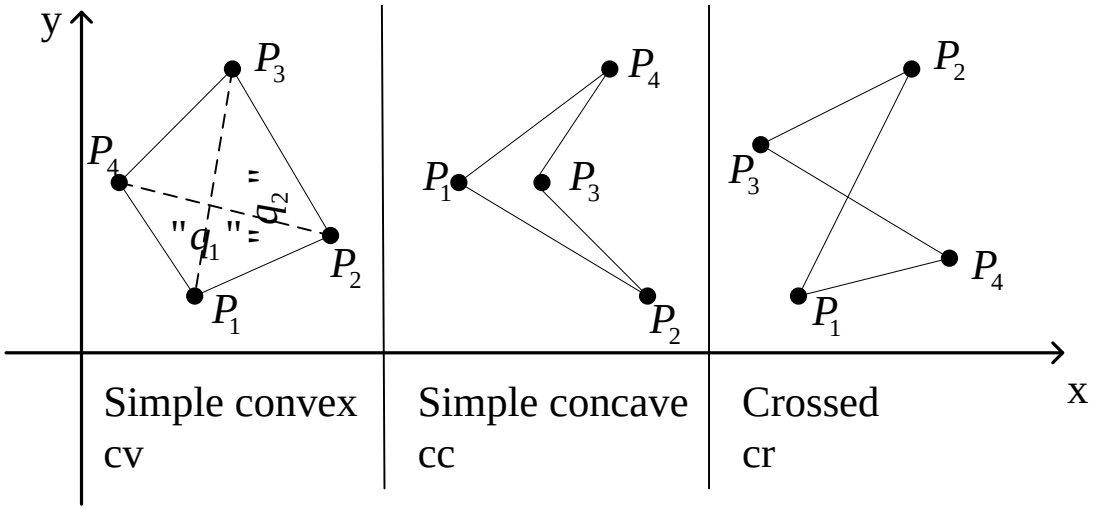
$$= \frac{1}{2}\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \end{vmatrix} - \frac{1}{2}\begin{vmatrix} x_1 & x_3 & x_2 \\ y_1 & y_3 & y_2 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} \qquad (12)$$

Equs (10) and (12) are identical, valid for cv and cc!
How about area $area_{\square}P_1P_2P_3P_4$ in the case cr?

But now a more sophisticated problem!

1.2.2 A quadrangle classification algorithm



With the “discriminators”.

$$q_1 = \frac{2\,area_{\triangle}P_1P_2P_4}{2\,area_{\square}P_1P_2P_3P_4} = \frac{\begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \tag{13.1}$$

$$q_2 = \frac{2\,area_{\triangle}P_1P_2P_3}{2\,area_{\square}P_1P_2P_3P_4} = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \tag{13.2}$$

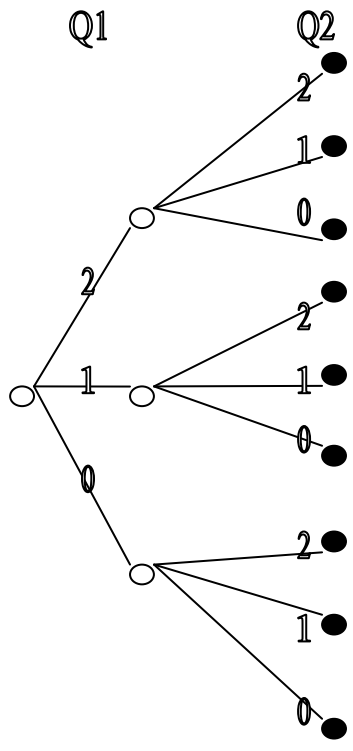
and the quantification encoding:

$$Q_i = \begin{cases} 0 & q_i < 0 \\ 1 \quad \text{if} & 0 < q_i < 1 \\ 2 & 1 < q_i \end{cases} \tag{13.3}$$

we get a very simple 2D quadrangle classification algorithm with an extensively inherent symmetry.

If		then $area_{\square} P_1P_2P_3P_4$ is:	
Q_1	Q_2		
1	1	convex (cv)	
0	1	concave (cc), the concave corner is located at point	P_1
1	0		P_2
2	1		P_3
1	2		P_4
0	2	crossed (cr), the crossing sides are:	$\overline{P_1P_2}$ and $\overline{P_3P_4}$
2	0		
0	0		$\overline{P_1P_4}$ and $\overline{P_2P_3}$
2	2		

Classification table.

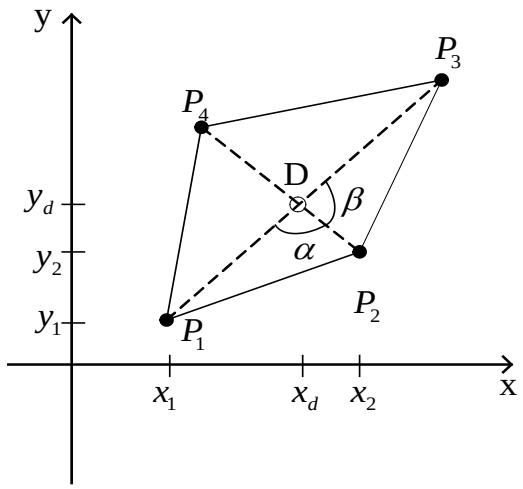


cr	$\overline{P_1P_4}$ and $\overline{P_2P_3}$
cc	P_3
cr	$\overline{P_1P_2}$ and $\overline{P_3P_4}$
cc	P_4
cv	
cc	P_2
cr	$\overline{P_1P_2}$ and $\overline{P_3P_4}$
cc	P_1
cr	$\overline{P_1P_4}$ and $\overline{P_2P_3}$

Classification ternary tree.
Describe the heuristic way from the problem to the result in detail.
Application: crossing detector in a sequence of connected line segments.
Much more sophisticated:

1.2.3 Angles and diagonal intersection point

Given a 2D simple convex quadrangle $\square P_1P_2P_3P_4$



by the coordinates of its vertices $P_i = (x_i, y_i)$; i=1,2,3,4

Find simple bdc expressions for:

1. the angles $\measuredangle P_4\underline{P_1}P_2$, $\measuredangle P_1\underline{P_2}P_3$, $\measuredangle P_2\underline{P_3}P_4$, $\measuredangle P_3\underline{P_4}P_1$,
2. the coordinates of the diagonal intersection point $D = (x_d, y_d)$,
3. the diagonal angles α and β.

Solution 1:

$$\tan\measuredangle P_4\underline{P_1}P_2 = \frac{\begin{vmatrix} x_4 & x_1 & x_2 \\ y_4 & y_1 & y_2 \end{vmatrix}}{\begin{vmatrix} x_4 & y_4 & x_1 & y_1 \\ y_1 & x_2 & y_2 & x_1 \end{vmatrix}} \qquad (14)$$

From $\tan\measuredangle P_4\underline{P_1}P_2$ to $\tan\measuredangle P_1\underline{P_2}P_3$ etc. via cyclic permutation!

$$\tan\measuredangle P_1\underline{P_2}P_3 = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_2 & y_2 \\ y_2 & x_3 & y_3 & x_2 \end{vmatrix}} \qquad (15)$$

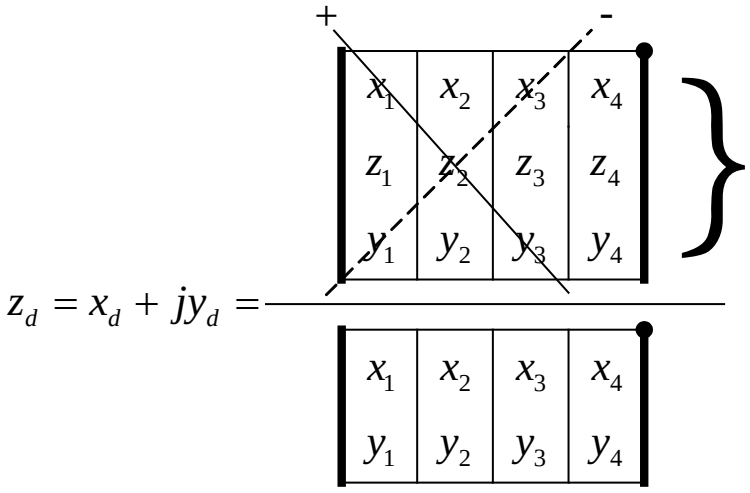
$$\tan\measuredangle P_2\underline{P_3}P_4 = \frac{\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 & x_3 & y_3 \\ y_3 & x_4 & y_4 & x_3 \end{vmatrix}} \qquad (16)$$

$$\tan\measuredangle P_3\underline{P_4}P_1 = \frac{\begin{vmatrix} x_3 & x_4 & x_1 \\ y_3 & y_4 & y_1 \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 & x_4 & y_4 \\ y_4 & x_1 & y_1 & x_4 \end{vmatrix}} \qquad (17)$$

Describe the way to these solutions in detail!

How about equs (14)-(17) in the cases cc and cr?

Solution 2:
Representation of D by a complex number:



Extension:
3 by 4 shortened determinant
=3 by 4 bdc!

$$z_d = x_d + jy_d = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \qquad (18)$$

Equ. (18) gives a convincing reflection of the given quadrangle, in contradiction to the following confusing 1D string notation!

$$z_d = \frac{x_1z_2y_3 + x_2z_3y_4 + x_3z_4y_1 + x_4z_1y_2 - x_1z_4y_3 - x_2z_1y_4 - x_3z_2y_1 - x_4z_3y_2}{x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_1y_4 - x_2y_1 - x_3y_2 - x_4y_3} \qquad (19)$$

Describe the way to this solution in detail!
Find a geometric interpretation for

$$\arg(z_d) = \arctan\frac{y_d}{x_d} \quad . \qquad (20)$$

Equ. (18) is valid for cv and cc, but how about the case cr?

<u>Solution</u> 3:

$$\tan\alpha = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix}} \qquad (21)$$

$$\tan\beta = -\tan\alpha = \frac{\begin{vmatrix} x_2 & x_3 & x_4 & x_1 \\ y_2 & y_3 & y_4 & y_1 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 & x_4 & y_4 \\ x_1 & x_3 & y_3 & y_1 \end{vmatrix}} \qquad (22)$$

Special cases:
a)

$$\alpha = \beta = \frac{\pi}{2} \Leftrightarrow \begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix} = 0 \ , \qquad (23)$$

that means a line through the points $P_1 = (x_1, y_1)$ and $P_3 = (x_3, y_3)$ and a line through the points $P_2 = (x_2, y_2)$ and $P_4 = (x_4, y_4)$ in 2D are mutually perpendicular iff:

$$\begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix} = 0 \ , \qquad (24)$$

and parallel iff

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = 0 \ , \qquad (25)$$

b) $P_4 := P_3 \ ; \ P_1 := P_4 \ \Leftrightarrow \ \measuredangle P_2 \underline{P_3} P_4 := \alpha \qquad (26)$

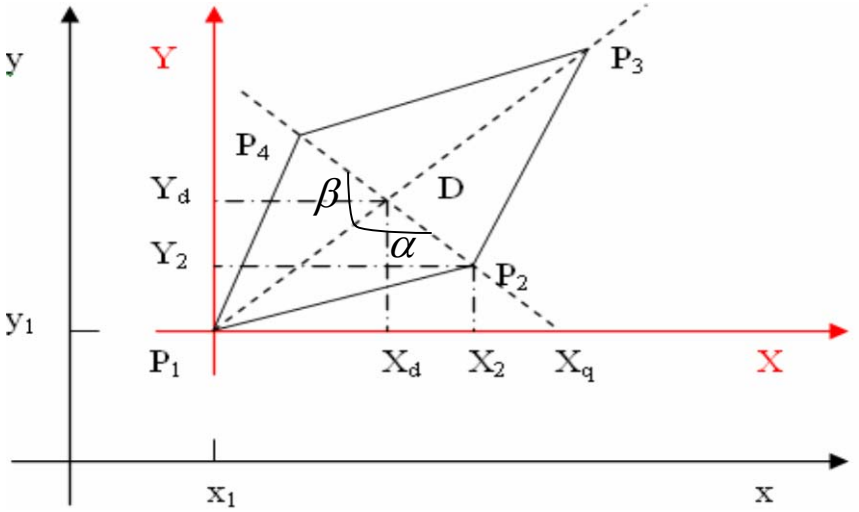
that means

$$\tan \measuredangle P_2 \underline{P_3} P_4 := \tan\alpha \ \text{ with } P_4 := P_3 \ \textit{and} \ P_1 := P_4 \qquad (27)$$

$$= \frac{\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 & x_3 & y_3 \\ x_3 & x_4 & y_4 & y_3 \end{vmatrix}}$$

Describe the way to equ. (21) in detail!

Solution:



We solve the given problem in the simplest way using a new coordinate system X – Y, connecting with the x-y-coordinate system by translation:

$X = x - x_1$

$Y = y - y_1$ (28)

That means: P_1 lies now in the origin of the new coordinate system, thus are all calculations simpler.

Angles in fig. $angle(P_3 P_1 Xaxis) = \lambda$; $angle(P_1 \underline{Xaxis} P_4) = \mu$ (29)

$\triangle P_1 Xaxis D$: $\alpha + \lambda + \mu = \pi \rightarrow \alpha = \pi - (\lambda + \mu)$ (30)

Trig. funct.: $$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$$ (31)

(30) and (31): $$\tan\alpha = \frac{\tan\pi - \tan(\lambda + \mu)}{1 + \tan\pi * \tan(\lambda + \mu)}$$ (32)

Trig. funct.: $\tan\pi = 0$ (33)

(33), (32) and (31): $\tan\alpha = -\tan(\lambda+\mu) = \dfrac{\tan\lambda + \tan\mu}{1 - \tan\lambda * \tan\mu}$ (34)

$\triangle P_3P_1Xaxis = \lambda$: $\tan\lambda = \dfrac{Y_3}{X_3}$ (35)

$\triangle P_1XasisP_4$: $\tan\mu = \dfrac{Y_4 - Y_2}{X_2 - X_4}$ (36)

(35), (36) and (34):

$$\tan\alpha = -\frac{\frac{Y_3}{X_3} + \frac{Y_4 - Y_2}{X_2 - X_4}}{1 - \frac{Y_3}{X_3} * \frac{Y_4 - Y_2}{X_2 - X_4}} = \frac{(X_2 - X_4)Y_3 + X_3(Y_4 - Y_2)}{X_3(X_4 - X_2) + (Y_4 - Y_2)Y_3} = \frac{X_2Y_3 + X_3Y_4 - X_3Y_2 - X_4Y_3}{X_3X_4 - X_2X_3 + Y_3Y_4 - Y_2Y_3} \quad (37)$$

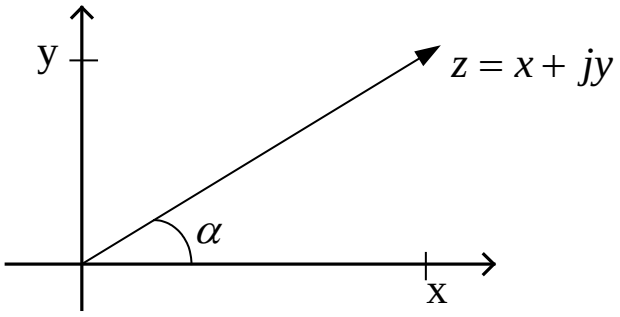
(37) in shortened determinant notation:

$$\tan\alpha = \frac{\begin{vmatrix} 0 & X_2 & X_3 & X_4 \\ 0 & Y_2 & Y_3 & Y_4 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & X_3 & Y_3 \\ Y_4 & X_2 & Y_2 & X_4 \end{vmatrix}} \quad (38)$$

(28) in (38) (bdc cross word puzzle principle: here resubstitute (0,0) by P_1) →

$$\tan\alpha = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix}} \quad (39)$$

Complex number interpretation of $\tan\alpha$



$$z = x + jy \tag{40.1}$$

$$; \quad \arg(z) = \alpha = \arctan\frac{y}{x} \tag{40.2}$$

$$\tan\alpha = \frac{y}{x} \tag{40.3}$$

$$|z| = \sqrt{x^2 + y^2} \tag{40.4}$$

(39) and (40) suggest the introduction of a complex number:

$$z = \begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix} + j \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = ? + 2 area\square P_1P_2P_3P_4 \tag{41.1}$$

$$= \overline{z_1}z_2 - z_2\overline{z_3} + \overline{z_3}z_4 - z_4\overline{z_1} = (\overline{z_1} - \overline{z_3})(z_2 - z_4) \tag{41.2}$$

$$= \begin{vmatrix} \overline{z_1} & 0 & \overline{z_3} & 0 \\ 0 & z_2 & 0 & z_4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \overline{z_1} & z_1 & \overline{z_3} & z_3 \\ \overline{z_4} & z_2 & \overline{z_2} & z_4 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} \overline{z_1} & \overline{z_2} & \overline{z_3} & \overline{z_4} \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \tag{41.3}$$

Discuss the location of the point z in relation to $\square P_1P_2P_3P_4$ for cv, cc and cr.

1.2.4 Centroid (mass center) $C = (x_C, y_C)$

Representation of C by a complex number:

$$z_c = x_c + jy_c = \frac{\begin{vmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_3 & x_4 & x_4 \\ z_1 & z_1 & z_2 & z_2 & z_3 & z_3 & z_4 & z_4 \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 & y_4 & y_4 \end{vmatrix}}{3\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \tag{42}$$

3 by 8 bdc!

Note: in general is $C \neq D$!

$$3z_c + z_d = z_1 + z_2 + z_3 + z_4 \qquad (43)$$

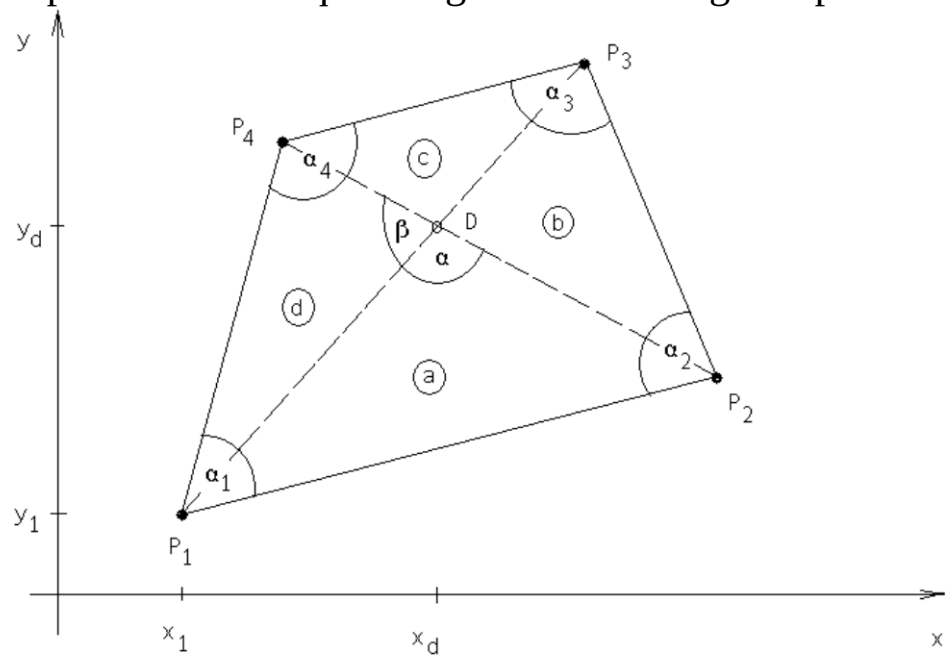
Equ. (42) gives a convincing notation for z_c in contradiction to the corresponding string notation (see equs (18) and (19)).
Describe the way to this solution in detail!

Find a connection between the angles $\arctan\frac{y_c}{x_c}$ and $\arctan\frac{y_d}{x_d}$.

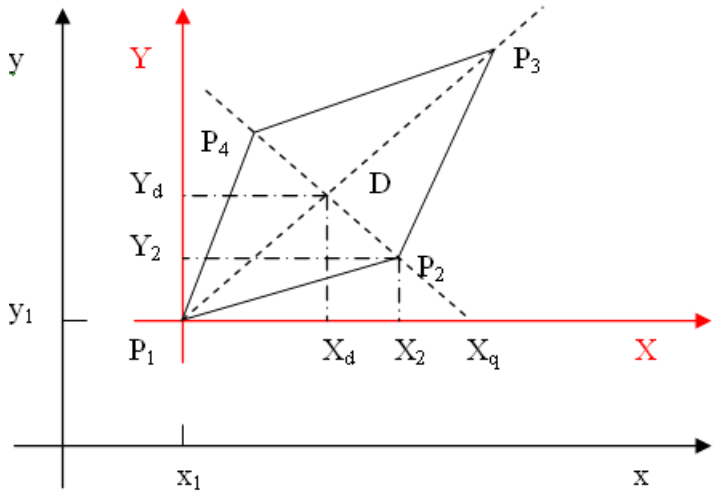
Equ. (42) is valid for cv and cc, how about cr?

1.2.5 Exercises to 1.2
1.2.5.1 Exercise 1

Find symmetric, simple and impressive bdc representations for the areas of the triangles $\triangle P_1P_2D$, $\triangle P_2P_3D$, $\triangle P_3P_4D$, $\triangle P_4P_1D$ in a simple convex 2D quadrangle with the diagonal point D



Solution:



We solve the given problem in the simplest way using a new coordinate system X – Y, connecting with the x-y-coordinate system by translation:

$X = x - x_1$

$Y = y - y_1$ (44)

That means: P_1 lies now in the origin of the new coordinate system, thus are all calculations simpler.

- $A(\Delta P_1P_2D) = A(\Delta P_1QD) - A(\Delta P_1QP_2) = \frac{1}{2}X_q(Y_d - Y_2)$ (45)
- $line(P_2P_4): \frac{Y - Y_2}{X - X_2} = \frac{Y_4 - Y_2}{X_4 - Y_2}$

  intersects the X-axis in $(X_q, 0) \rightarrow$

  $$X_q = \frac{X_2Y_4 - X_4Y_2}{Y_4 - Y_2} \quad (46)$$
- $line(P_2P_4): \frac{Y - Y_2}{X - X_2} = \frac{Y_4 - Y_2}{X_4 - X_2}$ intersects

  $line(P_1P_3): \frac{Y}{X} = \frac{Y_3}{X_3}$ in $(X_d, Y_d) \rightarrow$

  $$Y_d = \frac{(X_2Y_4 - X_4Y_2)Y_3}{X_2Y_3 + X_3Y_4 - X_3Y_2 - X_4Y_3} \quad (47)$$

  and

  $$Y_d - Y_2 = \frac{X_2Y_3Y_4 - \cancel{X_4Y_2Y_3} - X_2Y_2Y_3 - X_3Y_2Y_4 + X_2Y_2^2 + \cancel{X_4Y_2Y_3}}{X_2Y_3 + X_3Y_4 - X_3Y_4 - X_4Y_3}$$

  $$= \frac{(X_2Y_3 - X_3Y_2)(Y_4 - Y_2)}{X_2Y_3 + X_3Y_4 - X_3Y_4 - X_4Y_3} \quad (48)$$

- (46) and (48) in (45) →

$$A(\Delta P_1P_2D)=\frac{1}{2}\frac{X_2Y_4-X_4Y_2}{\cancel{Y_4-Y_2}}\cdot\frac{(X_2Y_3-X_3Y_2)\cancel{(Y_4-Y_2)}}{X_2Y_3+X_3Y_4-X_3Y_2-X_4Y_3}$$

$$=\frac{1}{2}\frac{\begin{vmatrix}0 & X_2 & X_3\\ 0 & Y_2 & Y_3\end{vmatrix}\begin{vmatrix}0 & X_2 & X_4\\ 0 & Y_2 & Y_4\end{vmatrix}}{\begin{vmatrix}0 & X_2 & X_3 & X_4\\ 0 & Y_2 & Y_3 & Y_4\end{vmatrix}} \qquad (49)$$

- (44) in (49) → shortened determinant representation (bdc cross word puzzle principle!):

$$A(\Delta P_1P_2D)=\frac{\frac{1}{2}\begin{vmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}\frac{1}{2}\begin{vmatrix}x_1 & x_2 & x_4\\ y_1 & y_2 & y_4\end{vmatrix}}{\frac{1}{2}\begin{vmatrix}x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\end{vmatrix}}$$

$$=\frac{A(\Delta P_1P_2P_3)\cdot A(\Delta P_1P_2P_4)}{A(P_1P_2P_3P_4)} \qquad (50)$$

- By cyclic permutation:

$$A(\Delta P_2P_3D)=\frac{\frac{1}{2}\begin{vmatrix}x_2 & x_3 & x_4\\ y_2 & y_3 & y_4\end{vmatrix}\frac{1}{2}\begin{vmatrix}x_2 & x_3 & x_1\\ y_2 & y_3 & y_1\end{vmatrix}}{\frac{1}{2}\begin{vmatrix}x_2 & x_3 & x_4 & x_1\\ y_2 & y_3 & y_4 & y_1\end{vmatrix}}$$

$$=\frac{A(\Delta P_2P_3P_4)\cdot A(\Delta P_2P_3P_1)}{A(P_2P_3P_4P_1)}$$

$$= \frac{A(\Delta P_1 P_2 P_3) \cdot A(\Delta P_2 P_3 P_4)}{A(P_1 P_2 P_3 P_4)} \tag{51}$$

$$A(\Delta P_3 P_4 D) = \frac{A(\Delta P_1 P_3 P_4) \cdot A(\Delta P_2 P_3 P_4)}{A(P_1 P_2 P_3 P_4)} \tag{52}$$

$$A(\Delta P_4 P_1 D) = \frac{A(\Delta P_1 P_2 P_3) \cdot A(\Delta P_1 P_3 P_4)}{A(P_1 P_2 P_3 P_4)}. \tag{53}$$

1.2.5.2 Exercise 2

Equ. (41) generates from the given vertices $P_i = (x_i, y_i)$; i=1(1)4 of a quadrangle $\square P_1 P_2 P_3 P_4$ a new point $P_5 = (x_5, y_5)$, in complex number notation

$$z_5 = \begin{vmatrix} \overline{z_1} & 0 & \overline{z_3} & 0 \\ 0 & z_2 & 0 & z_4 \end{vmatrix} \tag{54}$$

It is obviously to repeat this step, now starting with the vertices P_2, P_3, P_4 and P_5 of a new quadrangle $\square P_2 P_3 P_4 P_5$ and to continue this process by recursion

$$z_{i+4} = f(z_i, z_{i+1}, z_{i+2}, z_{i+3}) = \begin{vmatrix} \overline{z_i} & 0 & \overline{z_{i+2}} & 0 \\ 0 & z_{i+1} & 0 & z_{i+3} \end{vmatrix} \tag{55.1}$$

in a counting loop
i=1(1)n (55.2)

In this way we can generate and display
- a sequence of quadrangles $\square P_1 P_2 P_3 P_4$, $\square P_2 P_3 P_4 P_5$, $\square P_3 P_4 P_5 P_6$,… moving in 2D
- a 2D sequence of connected line segments $\overline{P_1 P_2}$, $\overline{P_2 P_3}$, $\overline{P_3 P_4}$, $\overline{P_4 P_5}$, $\overline{P_5 P_6}$,…

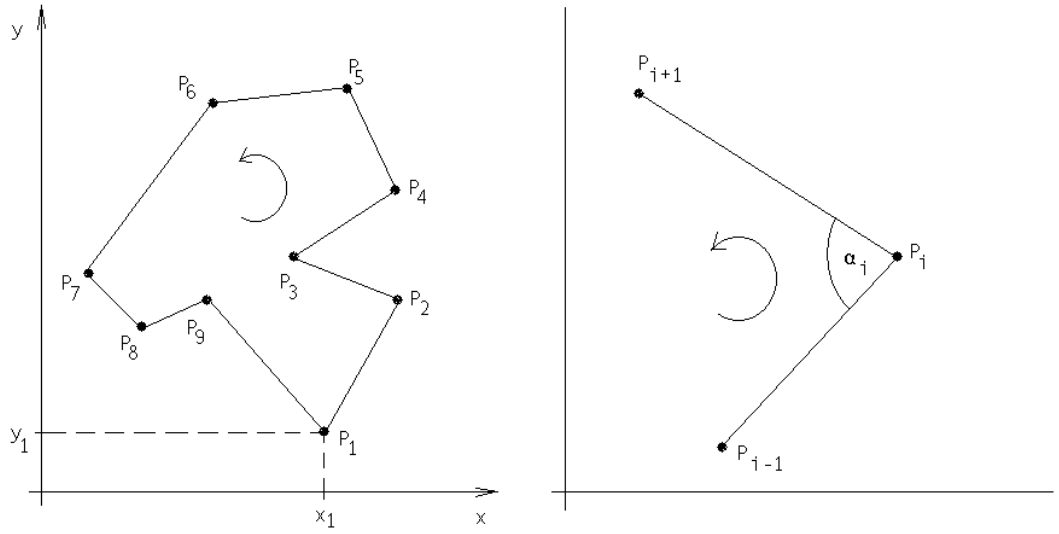
both depending from the initiate quadrangle $\square P_1 P_2 P_3 P_4$.
Inserting a suitable control function c(i) into the recursion equation

$$z_{i+4} = f(c(i),\ z_i, z_{i+1}, z_{i+2}, z_{i+3});\ i = 1(1)n \qquad (56)$$

we get a flexible graphic pattern generation system.
Test the performance of this system

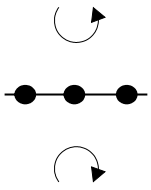
- to generate artistic computer graphics,
- to generate physical field pattern,
- to solve data mining problems.

1.3 Now from triangle and quadrangle to the simple non-crossed 2D polygon (n-gon)



1.3.1 Area and sum of the side squares

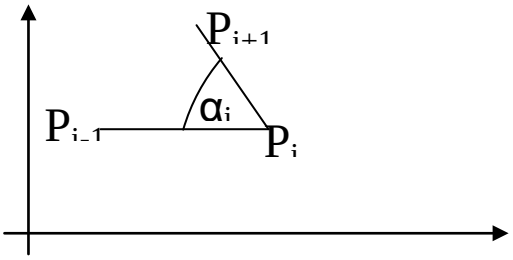
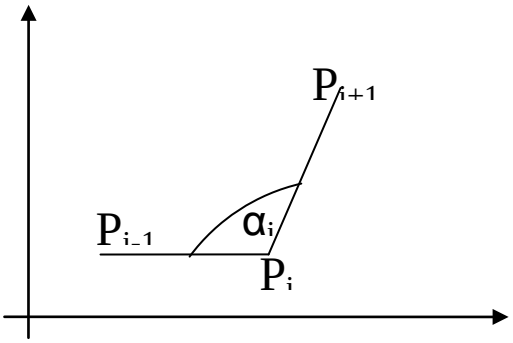
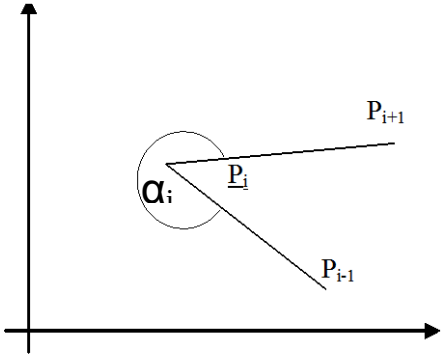
Convincing extensions of equs (52) and (54) are

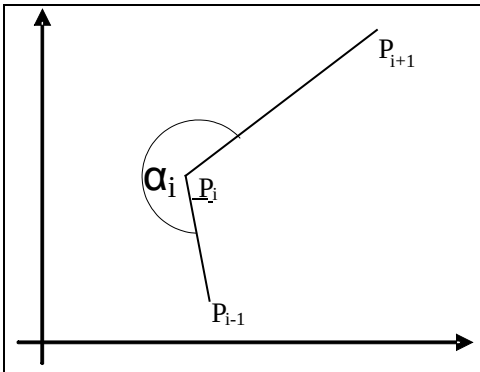
$$area\square P_1P_2P_3\cdots P_n = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & --- & x_n \\ y_1 & y_2 & y_3 & & y_n \end{vmatrix} \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff} \quad \text{  \& ?} \qquad (57)$$

See - E.W. Weisstein. Polygon Area.
http://mathworld.wolfram.com/PolygonArea.html
- http://www.DresdenAlgorithmicsChannel.de
Module 47 A Linear Genomic Algebra

$$\overline{P_1P_2}^2 + \overline{P_2P_3}^2 + \overline{P_3P_4}^2 + \cdots + \overline{P_nP_1}^2 = \begin{vmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & --- & x_n & y_n \\ y_n & x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & --- & x_n \end{vmatrix} \quad (58)$$

1.3.2 Angles

The vertex $P_{i-1}\ \underline{P_i}\ P_{i+1}$	is from outside	if and only if		$\alpha_i =$ $\measuredangle P_{i-1}\ \underline{P_i}\ P_{i+1}$
		N_i	D_i	
	convex, acute	>0	>0	$\beta_i =$ $\arctan\frac{\lvert N_i\rvert}{\lvert D_i\rvert}$ $0^0 < \alpha_i < 90^0$ $0^0 < \beta_i < 90^0$
	convex, obtuse	>0	<0	$180^0 - \beta_i$ $90^0 < \alpha_i < 180^0$
	concave, reflex acute	<0	>0	$360^0 - \beta_i$ $270^0 < \alpha_i < 360^0$

	concave, reflex obtuse	<0	<0	$180^0 + \beta_i$ $180^0 < \alpha_i < 270^0$

Classification table for α_i

$$N_i = \begin{vmatrix} x_{i-1} & x_i & x_{i+1} \\ y_{i-1} & y_i & y_{i+1} \end{vmatrix} \quad ; \qquad D_i = \begin{vmatrix} x_{i-1} & y_{i-1} & x_i & y_i \\ y_i & x_{i+1} & y_{i+1} & x_i \end{vmatrix} \qquad (59;\ 60)$$

1.3.2 Centroid (mass center)

Convincing extensions of equs (42) and (43):

$$z_c = \frac{1}{3} \frac{\begin{vmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_3 & --- & --- & x_n & x_n \\ z_1 & z_1 & z_2 & z_2 & z_3 & z_3 & --- & --- & z_n & z_n \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 & --- & --- & y_n & y_n \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & --- & x_n \\ y_1 & y_2 & y_3 & --- & y_n \end{vmatrix}} \qquad (61)$$

$$= \frac{1}{3}(z_1 + z_2 + z_3 \cdots + z_n) - \frac{1}{3} \frac{\begin{vmatrix} x_1 & x_2 & x_3 & --- & x_n \\ z_1 & z_2 & z_3 & --- & z_n \\ y_1 & y_2 & y_3 & --- & y_n \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & --- & x_n \\ y_1 & y_2 & y_3 & --- & y_n \end{vmatrix}} \qquad (62)$$

1.3.3 Extension of equ. (41): 4 $\rightarrow$ n=2k, cv and cr

$$z = x + jy = \begin{vmatrix} x_1 & y_1 & x_3 & y_3 & x_5 & y_5 & --- & x_{n-1} & y_{n-1} \\ y_n & x_2 & y_2 & x_4 & y_4 & x_6 & y_6 & --- & x_n \end{vmatrix}$$

$$+ j \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & --- & x_n \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & --- & y_n \end{vmatrix} \qquad (63.1)$$

$$= \begin{vmatrix} \overline{z_1} & 0 & \overline{z_3} & 0 & \overline{z_5} & --- & \overline{z_{n-1}} & 0 \\ 0 & z_2 & 0 & z_4 & 0 & & 0 & z_n \end{vmatrix} \qquad (63.2)$$

$$y = 2area\square P_1P_2P_3P_4P_5P_6\cdots P_n \ , \qquad (64)$$

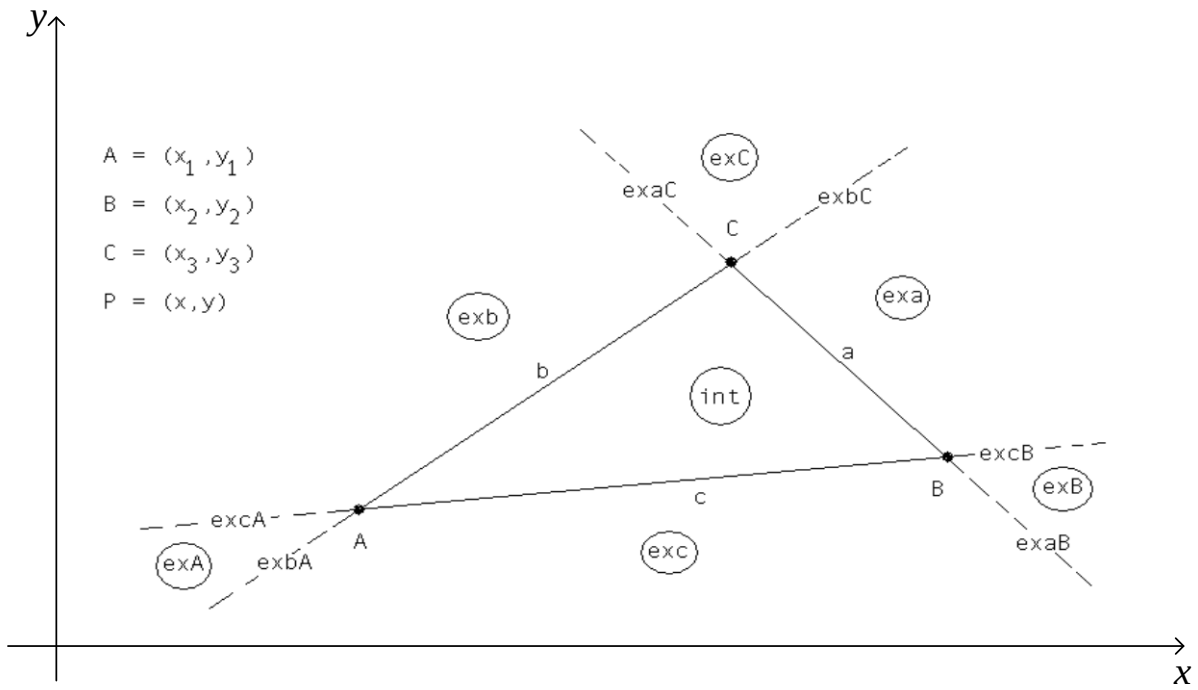
but what information about $\square P_1P_2P_3P_4P_5P_6\cdots P_n$ is encoded in x?
This is an open question.
2 partial answers:

- For n=4, cv&cc, x=0:
  the diagonal $\overline{P_1P_3}$ is perpendicular to the diagonal $\overline{P_2P_4}$.
- If $\square P_1P_2P_3P_4P_5P_6\cdots P_n$ regular and n=2k then x=0.

Find more.

1.4 Return to the triangle

1.4.1 A position code for the 2D constellation fixed triangle $\triangle ABC$ - movable point P



Given a fixed triangle $\triangle ABC$ by the coordinates of its vertices $A=(x_1,y_1);\ B=(x_2,y_2);\ C=(x_3,y_3)$ and a movable point $P=(x,y)$ in 2D.

$\triangle ABC$ defines $19=3^3-2^3$ distinguishable positions and position areas for P.

Find a position code $\triangle ABC$ - P using single bdc expressions.

Solution:

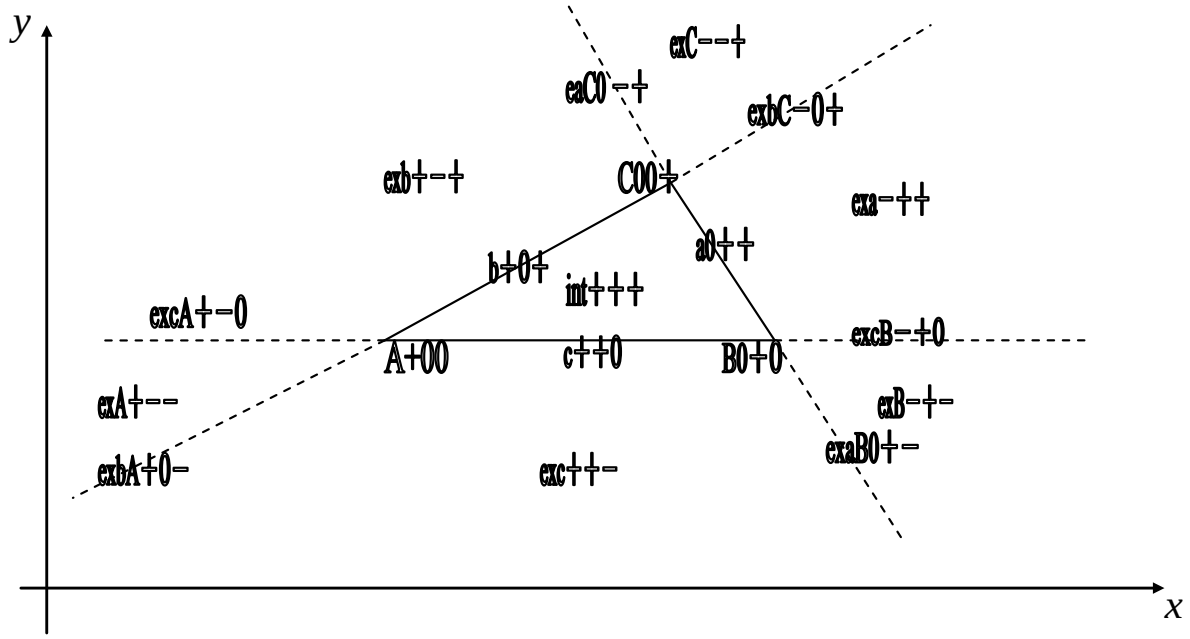
P=(x,y) is located at	if and only if $sgn(t_1)=$	$sgn(t_2)=$	$sgn(t_3)=$
int	+1	+1	+1
A	+1	0	0
B	0	+1	0
C	0	0	+1
a	0	+1	+1
b	+1	0	+1
c	+1	+1	0
exaB	0	+1	-1
exaC	0	-1	+1
exbA	+1	0	-1
exbC	-1	0	+1
excA	+1	-1	0
excB	-1	+1	0
exA	+1	-1	-1
exB	-1	+1	-1
exC	-1	-1	+1
exa	-1	+1	+1
exb	+1	-1	+1
exc	+1	+1	-1

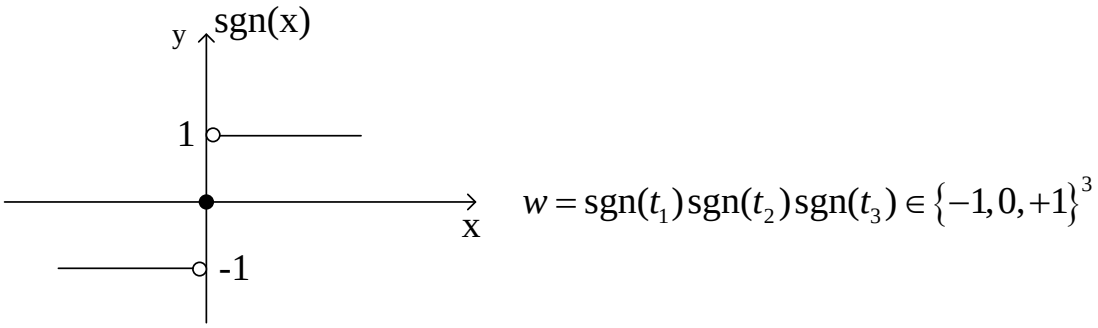
$$t_1=\frac{\begin{vmatrix} x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad (65.1)$$

$$t_2=\frac{\begin{vmatrix} x_1 & x & x_3 \\ y_1 & y & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad (65.2)$$

$$t_3=\frac{\begin{vmatrix} x_1 & x_2 & x \\ y_1 & y_2 & y \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad (65.3)$$

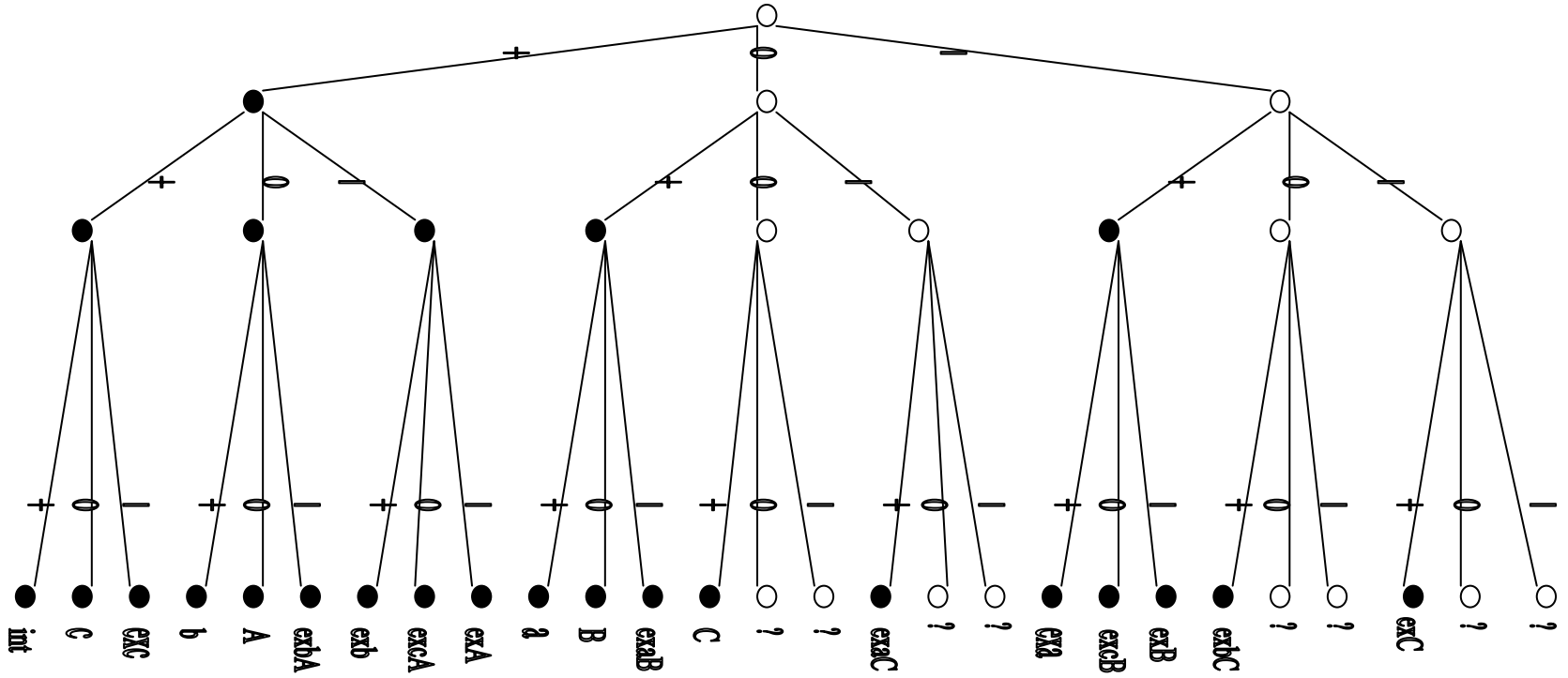
Coding table 1



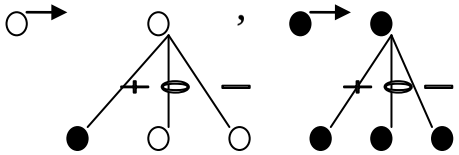


Coding table 2

Coding tree



Tree grammar: start symbol: o ; substitution rules:



The coding tree of the position code △ABC - P discloses the high inner symmetry of the presented solution.
The bdc expressions equs (45) can be compromised into the matrix equation

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{66}$$

$object \triangleq \triangle ABC$ — barycentric coordinates of P with respect to $\triangle ABC$ — image ≙ point P

mapping system ≤≡ parameters

the mapping equation of the position code △ABC - P.

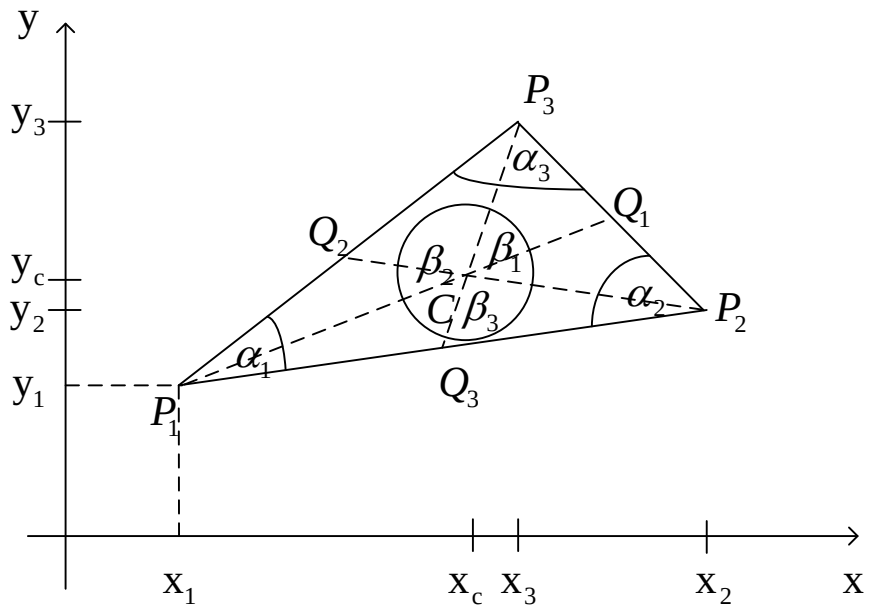
1.4.2 Exercises to 1.4

1.4.2.1 Exercise 1

Find the barycentric coordinates t_1, t_2, and t_3 of the four special points centroid, circumcenter, incenter and orthocenter of the triangle $\triangle P_1P_2P_3$ in function of its vertex coordinates (x_i, y_i); i=1(1)3, in function of its sides a, b, c and in function of its angles α_1, α_2, α_3.

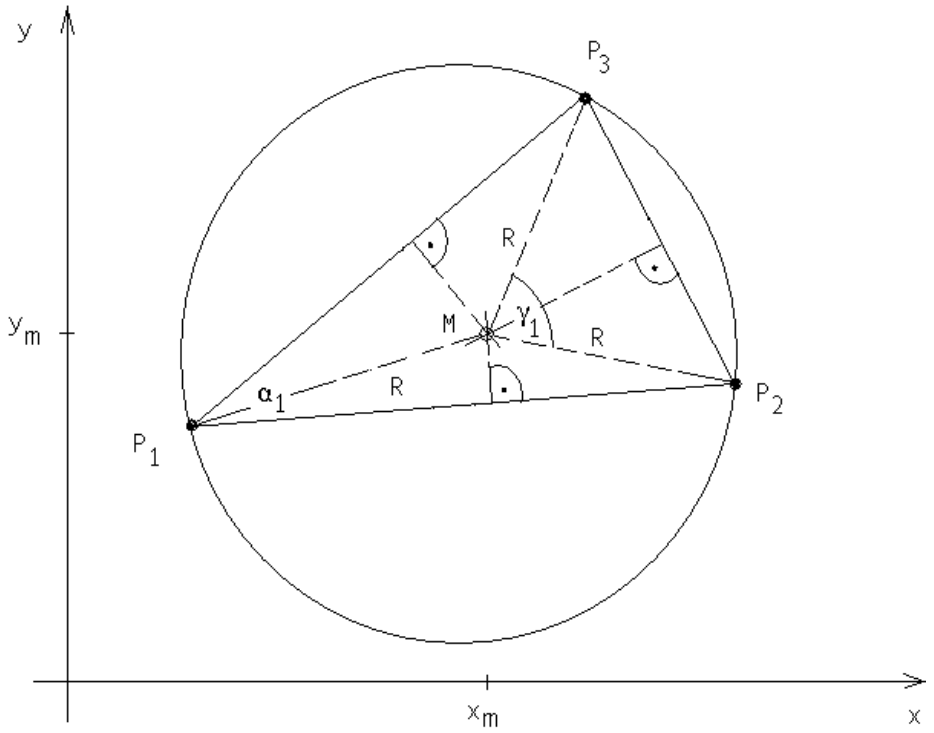
Solutions

1.4.2.1.1 Centroid C = intersection point of medians



$$t_1 = t_2 = t_3 = \frac{1}{3} \tag{67}$$

1.4.2.1.2 Circumcenter M = intersection point of the perpendicular bisectors of the sides



$$t_1 = \frac{1}{2} \frac{\begin{vmatrix} x_2 & y_2 & x_3 & y_3 \\ y_3 & x_2 & y_2 & x_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \cdot \frac{\begin{vmatrix} x_1 & y_1 & x_2 & y_2 \\ y_3 & x_1 & y_1 & x_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \tag{68.1}$$

$$= \frac{\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 1 & 1 \\ b^2 & 0 & 0 & 1 \\ c^2 & 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 1 & 1 \\ b^2 & 1 & 0 & 1 \\ c^2 & 1 & 1 & 0 \end{vmatrix}}; \text{ classical determinants} \tag{68.2}$$

$$= \frac{\sin 2\alpha_1}{\sin 2\alpha_1 + \sin 2\alpha_2 + \sin 2\alpha_3} \tag{68.3}$$

$$t_2 = \frac{1}{2} \frac{\begin{vmatrix} x_3 & y_3 & x_1 & y_1 \\ y_1 & x_3 & y_3 & x_1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \cdot \frac{\begin{vmatrix} x_2 & y_2 & x_3 & y_3 \\ y_1 & x_2 & y_2 & x_1 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \tag{69.1}$$

$$= \frac{\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 0 & 1 \\ b^2 & 1 & 0 & 1 \\ c^2 & 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 1 & 1 \\ b^2 & 1 & 0 & 1 \\ c^2 & 1 & 1 & 0 \end{vmatrix}} \tag{69.2}$$

$$= \frac{\sin 2\alpha_2}{\sin 2\alpha_1 + \sin 2\alpha_2 + \sin 2\alpha_3} \tag{69.3}$$

$$t_3 = \frac{1}{2} \frac{\begin{vmatrix} x_1 & y_1 & x_2 & y_2 \\ y_2 & x_1 & y_1 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \cdot \frac{\begin{vmatrix} x_3 & y_3 & x_1 & y_1 \\ y_2 & x_3 & y_3 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \tag{70.1}$$

$$= \frac{\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 1 & 0 \\ b^2 & 1 & 0 & 0 \\ c^2 & 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 1 & 1 \\ b^2 & 1 & 0 & 1 \\ c^2 & 1 & 1 & 0 \end{vmatrix}} \qquad (70.2)$$

$$= \frac{\sin 2\alpha_3}{\sin 2\alpha_1 + \sin 2\alpha_2 + \sin 2\alpha_3} \qquad (70.3)$$

1.4.2.1.3 Incenter = intersection point of the angle bisectors .

$$t_i = \frac{\sin \alpha_i}{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3} \;\; ; \;\; i = 1(1)3$$

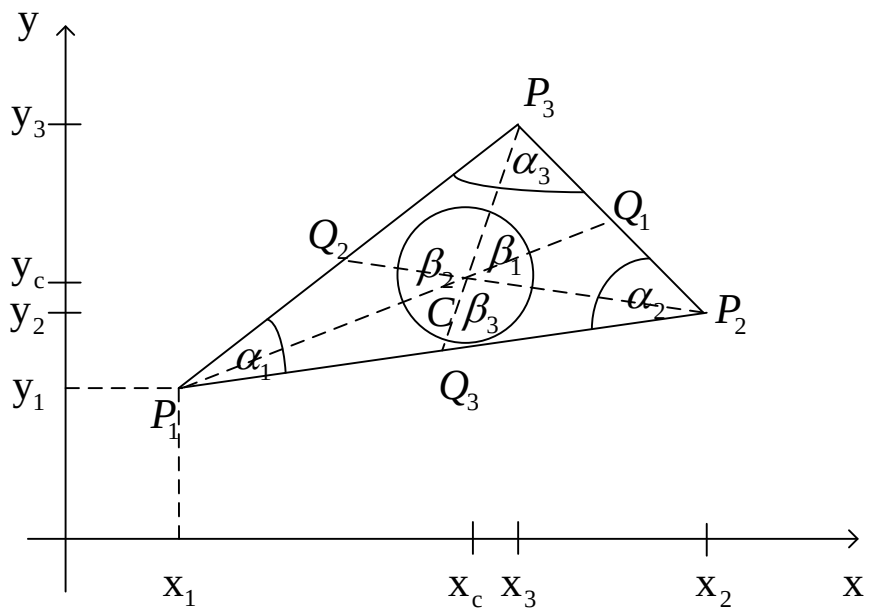
1.4.2.1.4 Orthocenter = intersection point of the altitudes

$$t_i = \frac{\tan \alpha_i}{\tan \alpha_1 + \tan \alpha_2 + \tan \alpha_3} \;\; ; \;\; i = 1(1)3$$

1.4.2.1.5 ? http://faculty.evansville.edu/ck6/tcenters/index.html

$$t_i = \frac{\cos \alpha_i}{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3} \;\; ; \;\; i = 1(1)3$$

1.4.2.2 Exercise 2



Find bdc expressions for the angles $\alpha_1,\ \alpha_2,\ \alpha_3$ and its centroid "patters" $\beta_1,\ \beta_2,\ \beta_3$ of the 2D triangle $\triangle P_1P_2P_3$ given by $P_i = (x_i, y_i)$; i=1(1)3.

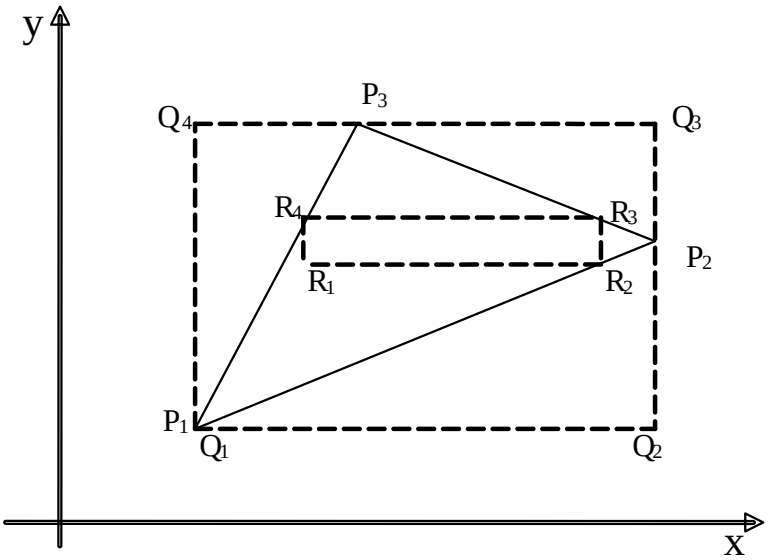
Solution:

$$tan\ \alpha_1 = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_2 & y_2 \\ y_3 & x_1 & y_1 & x_3 \end{vmatrix}} \tag{71}$$

$$tan\ \beta_1 = \frac{3\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{matrix} \underbrace{\begin{vmatrix} x_3 & y_3 & x_2 & y_2 \\ y_3 & x_2 & y_2 & x_3 \end{vmatrix}} & \underbrace{\begin{vmatrix} x_1 & y_1 & x_2 & y_2 \\ y_3 & x_1 & y_1 & x_3 \end{vmatrix}} & \underbrace{\begin{vmatrix} x_3 & y_3 & x_2 & y_2 \\ y_3 & x_2 & y_2 & x_3 \end{vmatrix}} \\ \text{left wing} & \text{core} & \text{right wing} \end{matrix}} \tag{72}$$

1.4.2.3 Exercise 3

Given a 2D triangle $\triangle P_1P_2P_3$ by the coordinates of its vertices $P_i = (x_i, y_i)$; i=1(1)3.

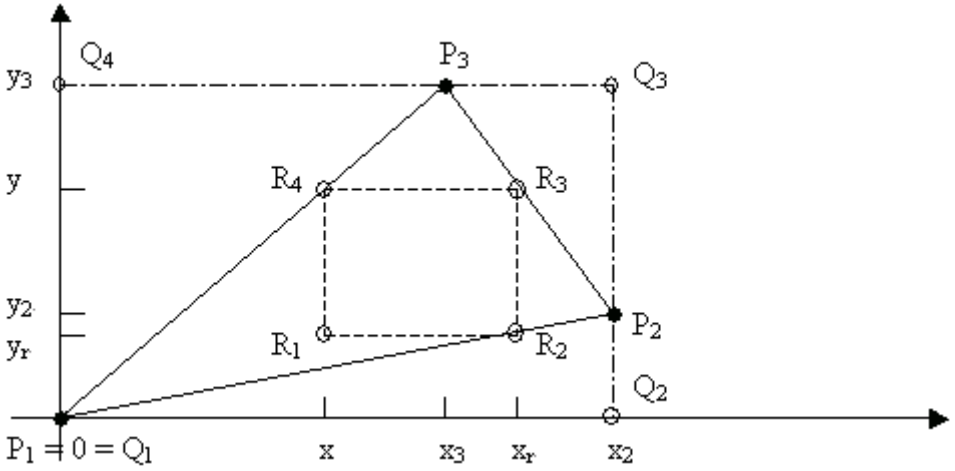


The rectangle $\square Q_1Q_2Q_3Q_4$ is the smallest axially parallel rectangle including the given triangle $\triangle P_1P_2P_3$.

- Find the vertex coordinates and the area of the largest axially parallel rectangle $\square R_1R_2R_3R_4$ includable into the given triangle $\triangle P_1P_2P_3$. Go the simplest way!
- Find the relation between A($\square Q_1Q_2Q_3Q_4$) and A($R_1R_2R_3R_4$).

Solutions

To solve the given problem in the simplest way but without restriction of validity, we translate the $\Delta P_1P_2P_3$ to the position $P_1 = (0,0)$; that means: P_1 lies in the origin of the x – y coordinate system, all calculations are so the simplest one.



- $A(\Delta P_1P_2P_3)=\frac{1}{2}(x_2y_3-x_3y_2)$ (74)

  $A(Q_1Q_2Q_3Q_4)=x_2y_3$ (75)

  $A(R_1R_2R_3R_4)=(x_r-x)(y-y_r)$ (76)

- $line(P_1P_2):\ y_r=\frac{y_2}{x_2}x_r$ (77)

  $line(P_1P_3):\ y=\frac{y_3}{x_3}x$ (78)

  $line(P_2P_3):\ \frac{y-y_2}{x_r-x_2}=\frac{y_3-y_2}{x_3-x_2}$ (79)

- (77) and (78) into (79) →

  $x_r-x=\frac{x_2y_3-x_3y_2}{x_3(y_3-y_2)}(x_3-x)$ (80)

  $y-y_r=\frac{x_2y_3-x_3y_2}{x_2x_3(y_3-y_2)}(xy_3-x_3y_2)$ (81)

- (80) and (81) into (76) →

  $A(R_1R_2R_3R_4)=f(x)$

  $f(x)=\underbrace{\frac{(x_2y_3-x_3y_2)^2}{x_2x_3^2(y_3-y_2)^2}}_{c>0}(x_3-x)(xy_3-x_3y_2)$ (82)

  $\frac{df(x)}{dx}=c((x_3-x)y_3-(xy_3-x_3y_2))=0$

  $\rightarrow x=\frac{1}{2}\frac{x_3}{y_3}(y_2+y_3)$ (83)

  $\frac{d^2f(x)}{dx^2}=-2cy_3<0\rightarrow Max.!$ (84)

- (83) with (84) into (82); (83) into (78), (80), (81) →

  $A(R_1R_2R_3R_4)_{\max}=(\frac{x_2y_3-x_3y_2}{2})^2\cdot\frac{1}{x_2y_3}$ (85)

  with

  $(x,y)_{\max}=(\frac{x_3}{y_3}\cdot\frac{y_2+y_3}{2},\frac{y_2+y_3}{2})$ (86)

$$(x_r, y_r)_{\max} = (\frac{x_2 + x_3}{2}, \frac{y_2}{x_2} \cdot \frac{x_2 + x_3}{2}) \quad (87)$$

- Result

$$R_1 = (x, y_r) = (\frac{x_3}{y_3} \cdot \frac{y_2 + y_3}{2}, \frac{y_2}{x_2} \cdot \frac{x_2 + x_3}{2}) \quad (88)$$

$$R_2 = (x_r, y_r) = (\frac{x_2 + y_3}{2}, \frac{y_2}{x_2} \cdot \frac{x_2 + x_3}{2}) \quad (89)$$

$$R_3 = (x_r, y_r) = (\frac{x_2 + y_3}{2}, \frac{y_2 + y_3}{2}) \quad (90)$$

$$R_4 = (x, y) = (\frac{x_3}{y_3} \cdot \frac{y_2 + y_3}{2}, \frac{y_2 + y_3}{2}) \quad (91)$$

$$A(R_1R_2R_3R_4)_{\max} = (\frac{x_2y_3 - x_3y_2}{2})^2 \cdot \frac{1}{x_2y_3} \quad (92)$$

(85) with (74) and (75) $\rightarrow$

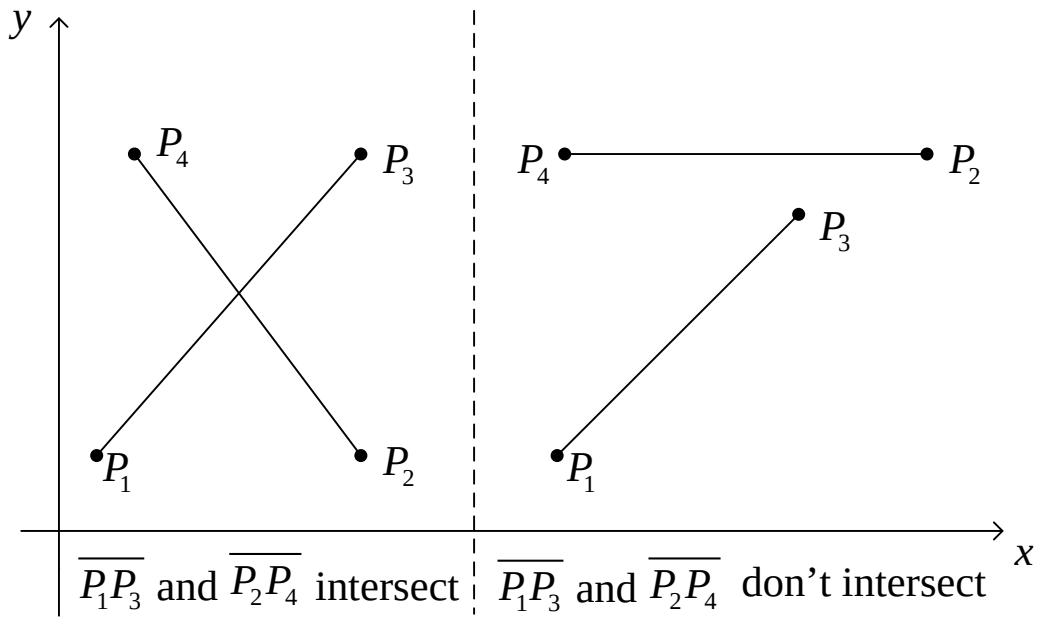
$$A(Q_1Q_2Q_3Q_4) \cdot A(R_1R_2R_3R_4)_{\max} = (A(\Delta P_1P_2P_3))^2 \rightarrow \quad (93)$$

$$\underline{A(\Delta P_1P_2P_3)} = \underline{\underline{g(A(Q_1Q_2Q_3Q_4), A(R_1R_2R_3R_4)_{\max}).}} \quad (94)$$

Find a simpler way to get this result: without differential calculus, only inside bdc linear algebra.

1.5 Advanced exercises

1.5.1 Advanced exercise 1



Verify and discuss!

The line segments $\overline{P_1P_3}$ and $\overline{P_2P_4}$ intersect in 2D if and only if

$$\frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_4 & x_3 \\ y_1 & y_4 & y_3 \end{vmatrix}} < 0 \quad \text{and} \quad \frac{\begin{vmatrix} x_2 & x_1 & x_3 \\ y_2 & y_1 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \end{vmatrix}} < 0 \tag{95}$$

in the point D=(x_d, y_d); $z_d = x_d + jy_d$

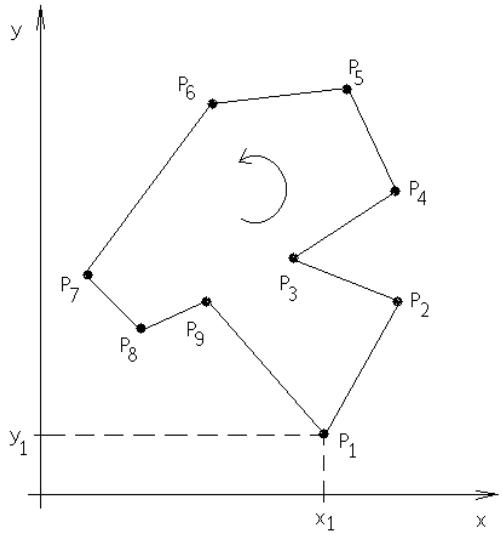
$$z_d = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \tag{96}$$

and form at D the angle $\measuredangle$

$$\tan\measuredangle = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix}} . \qquad (97)$$

1.5.2 Advanced exercise 2

A bridge building to calculus.



The contour integral around a simple 2D polygon $\square P_1P_2P_3P_4P_5P_6\cdots P_n$

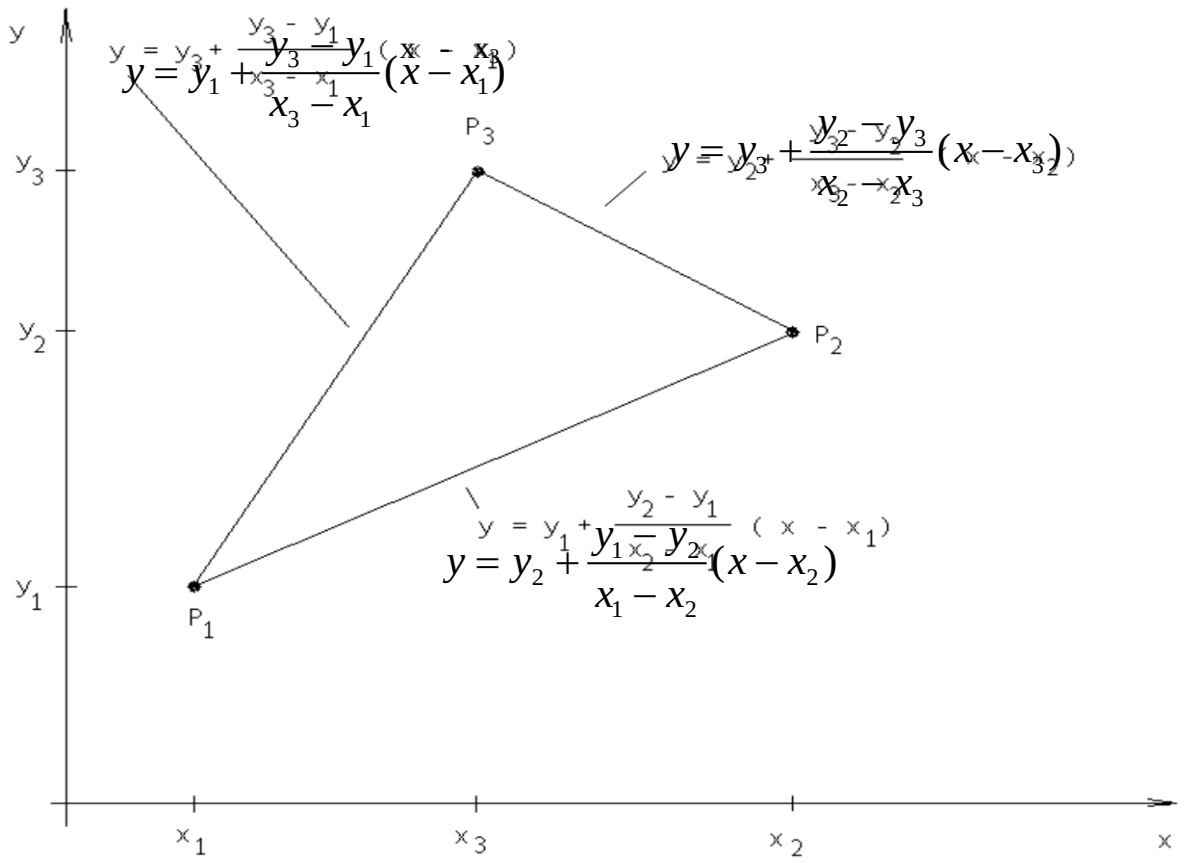
$$I(\mathrm{k},\mathrm{l},\mathrm{n}) = \oint_{\substack{\text{polygon} \\ P_1P_2P_3\cdots P_n}} \mathrm{x}^{\mathrm{k}} y(x)^l dx \qquad (98.1)$$

given by the coordinates of its vertices

$P_i = (x_i, y_i)$; i=1(1)n (98.2)

can be represented by a bdc.

Simple example: contour = $\triangle P_1P_2P_3$.



$$I(0,1,3) = \oint_{\triangle P_1P_2P_3} y(x)dx = \int_{x_1}^{x_3} (y_1 + \frac{y_3 - y_1}{x_3 - x_1}(x - x_1))dx$$

$$+ \int_{x_3}^{x_2} (y_3 + \frac{y_2 - y_3}{x_2 - x_3}(x - x_3))dx + \int_{x_2}^{x_1} (y_2 + \frac{y_1 - y_2}{x_1 - x_2}(x - x_2))dx$$

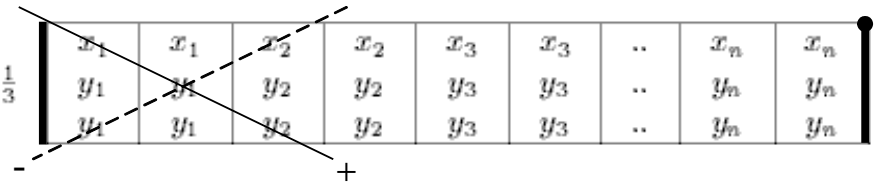
$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \text{area}\triangle P_1P_2P_3 \qquad (99)$$

$$I(0,1,n) = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & .. & x_n \\ y_1 & y_2 & y_3 & .. & y_n \end{vmatrix} \qquad (100)$$

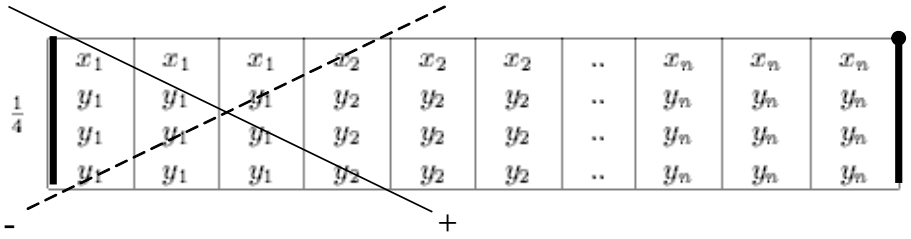
\- +

$$I(0,2,n) = \frac{1}{3}$$



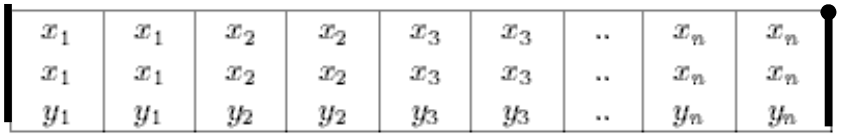
(101)

$$I(0,3,n) = \frac{1}{4}$$



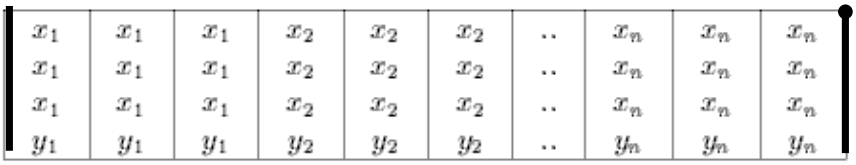
(102)

$$I(1,1,n) = \frac{1}{6}$$



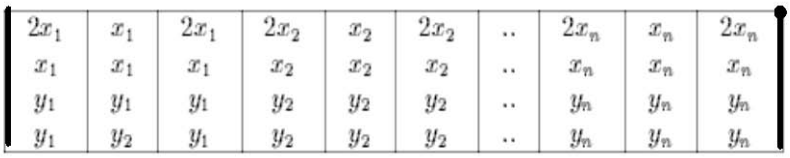
(103)

$$I(2,1,n) = \frac{1}{12}$$



(104)

$$I(1,2,n) = \frac{1}{12}$$



(105)

Applications regarding polygon $P_1P_2P_3P_4P_5P_6\cdots P_n$

- Area

  I(0,1,n) = area polygon $P_1P_2P_3P_4P_5P_6\cdots P_n$ (106)

- Centroid C = (x_c, y_c)

$$\frac{I(1,1,n)}{I(0,1,n)} = x_c;\ \frac{I(0,2,n)}{2I(0,1,n)} = y_c \quad (107)$$

- Solid of revolution generated by revolving the above 2D polygon around the x-axis
  - Volume

$$\pi I(0,2,n) = Vrotx\ P_1P_2P_3P_4P_5P_6\cdots P_n \quad (108)$$

  - Centroid Crotx $P_1P_2P_3P_4P_5P_6\cdots P_n = (x_{cx}, y_{cx}, z_{cx})$

$$\frac{I(1,2,n)}{I(0,2,n)} = x_c; \quad y_{cx} = z_{cx} = 0 \tag{109}$$

Find more!

How about stretcher bond

x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4
y_4	x_1	y_1	x_2	y_2	x_3	y_3	x_4

;

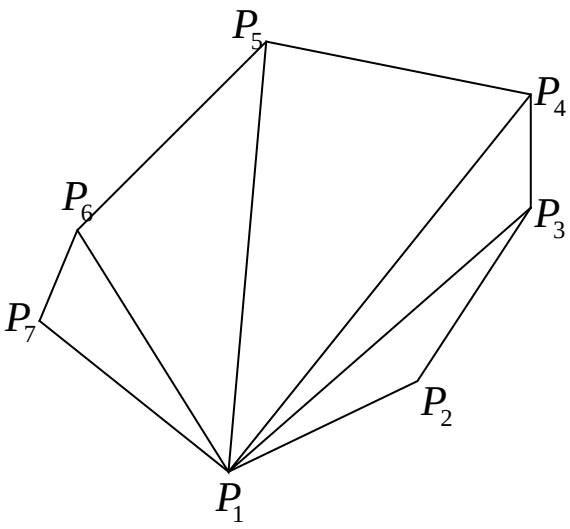
x_1	y_1	x_3	y_3
y_4	x_2	y_2	x_4

and integral?

1.5.3 Advanced exercise 3

A bridge building to pattern recognition/ image processing/ feature extraction

The calculation of the area of a simple 2D polygon (n-gon).
$\square P_1P_2P_3\cdots P_n$; $P_i=(x_i, y_i)$; i=1(1)n, n≥4
via triangulation, f.e.



$$area\square P_1P_2P_3P_4P_5P_6P_7 = area\triangle P_1P_2P_3 + area\triangle P_1P_3P_4 + area\triangle P_1P_4P_5 + area\triangle P_1P_5P_6 + area\triangle P_1P_6P_7 \tag{110}$$

integrates only neighbouring points:

$$area\square P_1P_2P_3P_4P_5P_6P_7 = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{vmatrix} =$$

$$= \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \left(\begin{vmatrix} x_4 & x_5 \\ y_4 & y_5 \end{vmatrix} + \begin{vmatrix} x_5 & x_6 \\ y_5 & y_6 \end{vmatrix} + \begin{vmatrix} x_6 & x_7 \\ y_6 & y_7 \end{vmatrix} + \begin{vmatrix} x_7 & x_1 \\ y_7 & y_1 \end{vmatrix} \right. \right. \tag{111}$$

and provides so only information about the size, but no information about the shape of the n-gon.

A simple way to include this information about the shape is to divide the given n-gon

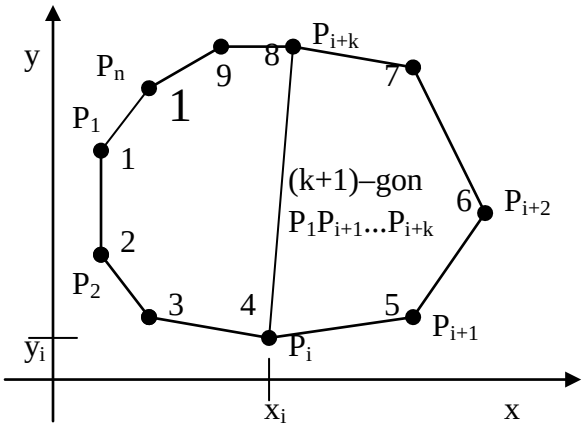
$\square P_1P_2P_3\cdots P_n$; $P_i=(x_i, y_i)$; i=1(1)n, n≥4

into each n (k+1)-subgons

$\square P_iP_{i+1}P_{i+2}\cdots P_{i+k}$; i=1(1)n

k=2(1)n-2

$P_m = P_{m-n}$ for m ≥ n



with the normalized areas:

$$\frac{area\square P_iP_{i+1}P_{i+2}\cdots P_{i+k}}{area\square P_1P_2P_3P_4P_5P_6P_7} = \frac{\begin{vmatrix} x_i & x_{i+1} & x_{i+2} & --- & x_{i+k} \\ y_i & y_{i+1} & y_{i+2} & --- & y_{i+k} \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & --- & x_n \\ y_1 & y_2 & y_3 & --- & y_n \end{vmatrix}} \quad . \tag{112}$$

The vector:

$$\vec{a} = \begin{bmatrix} a(2;\ P_1P_2P_3...P_n) \\ a(3;\ P_1P_2P_3...P_n) \\ a(4;\ P_1P_2P_3...P_n) \\ : \\ a(n-2;\ P_1P_2P_3...P_n) \end{bmatrix}$$

of the n-3 sums of this normalized areas:

$$a(k;\ P_1P_2P_3\cdots P_n) = \frac{\sum_{i=1}^{n} area\square P_iP_{i+1}P_{i+2}\cdots P_{i+k}}{area\square P_1P_2P_3\cdots P_n} - k \tag{113}$$

characterizes then the shape of $\square P_1P_2P_3\cdots P_n$.

• Find a simple, a blue ocean representation of equ. (113) by 2 by n bdc.

Solving advice:
For n=7 we have:

$$a(2, P_1\cdots P_7) = \frac{area\square P_1P_3P_5P_7P_2P_4P_6}{area\square P_1P_2P_3P_4P_5P_6P_7}$$

$$= \frac{\begin{vmatrix} x_1 & x_3 & x_5 & x_7 & x_2 & x_4 & x_6 \\ y_1 & y_3 & y_5 & y_7 & y_2 & y_4 & y_6 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{vmatrix}} \tag{114}$$

$$a(3, P_1\cdots P_7) = \frac{area\square P_1P_4P_7P_3P_6P_2P_5}{area\square P_1P_2P_3P_4P_5P_6P_7} \tag{115}$$

$$a((k=3), P_1\cdots P_{(n=7)}) = \frac{area\square P_1P_4P_7P_{\substack{(pos(i,k,n)=pos(4,3,7)=3) \\ =(i-1)k+1 \bmod n}} P_6P_2P_5}{area\square P_1P_2P_3P_4P_5P_6P_7} \tag{116}$$

$$a(4, P_1 \cdots P_7) = \frac{area\square P_1P_5P_2P_6P_3P_7P_4}{area\square P_1P_2P_3P_4P_5P_6P_7} \tag{117}$$

$$a(5, P_1 \cdots P_7) = \frac{area\square P_1P_6P_4P_2P_7P_5P_3}{area\square P_1P_2P_3P_4P_5P_6P_7} \tag{118}$$

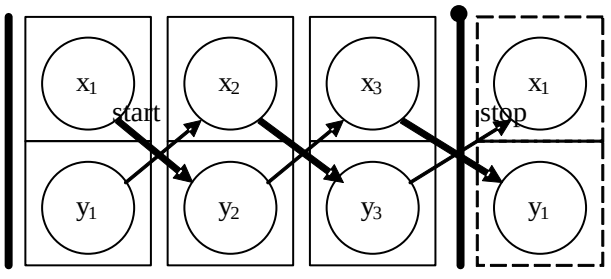
- Using this result, show that a(k; $P_1P_2P_3...P_n$) is invariant against translation, rotation, and scaling of the n-gon.
- Examine experimentally the "$\vec{a}$-spectra" of characteristic polygons. Discuss on the basis of your results the practical applicability of the "$\vec{a}$-spectrum" as a feature extraction tool.
- How about the invariance of the "$\vec{a}$-spectrum" against flipping, affine transformations, and elastic deformations of the polygon.
- How about the robustness against distortions, occlusions, and noise?
- Is it possible to reconstruct the polygon $P_1P_2P_3 \cdots P_n$ from $\vec{a}$?

1.5.4 Advanced exercise 4

Sarrus operation and navigation rule

The main idea of part 1.1 was to shorten the classical 3 by 3 determinant representation of the area of a 2D triangle $\triangle P_1P_2P_3$ in function of its vertex coordinates (x_1, y_1); i = 1(1)3, by cutting the first row containing only redundancy and to embed the resulting 2 by 3 determinant into a 2 by 3 data cylinder.
The bdc representation

$$area\triangle P_1P_2P_3 = \frac{1}{2}$$



$$= \frac{1}{2}(+x_1 * y_2 - y_2 * x_3 + x_3 * y_1 - y_1 * x_2 + x_2 * y_3 - y_3 * x_1)$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3) \tag{119}$$

includes in detail:

• the data array, here data cylinder with 2 traces and 3 segments (shortened 2 by 3determinant),
• the input data, here $x_1, y_1, x_2, y_2, x_3, y_3,$
• the local operator, here Sarrus operator

$$\text{a} \longrightarrow \text{b} = +\text{a*b} \;; \qquad \text{a} \longrightarrow \text{b} = -\text{a*b}$$

• and the instruction for the navigation of the local operator through the data array, here Sarrus navigation (see figure).

It is obvious to investigate whether it is possible to compress the classical 4 by 4 determinant representation of the area of a triangle $\triangle P_1P_2P_3$ in function of its side length $a = \overline{P_2P_3},\ \text{b} = \overline{P_3P_1},\ \text{c} = \overline{P_1P_2}$

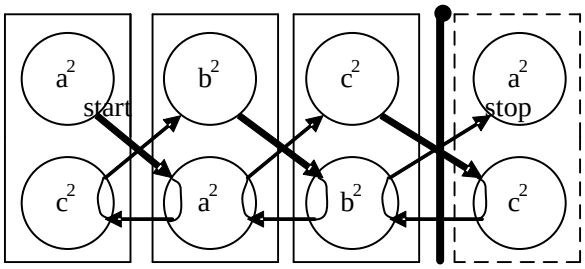
$$(area_{\triangle P_1P_2P_3})^2 = -\frac{1}{16}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} = -\frac{1}{16}\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & 1 & 1 \\ b^2 & 1 & 0 & 1 \\ c^2 & 1 & 1 & 0 \end{vmatrix}$$

$$= -\frac{1}{16}(a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)) \qquad (120)$$

in a similar and also extensible way.

*bdc Solution

$$(area_{\triangle P_1P_2P_3})^2 = -\frac{1}{16}\left|\ \right.$$



$$= -\frac{1}{16}(+a^2 * a^2 - a^2 * c^2 - c^2 * b^2 + b^2 * b^2 - b^2 * c^2 -$$
$$a^2 * c^2 + c^2 * c^2 - c^2 * b^2 - b^2 * a^2)$$
$$= -\frac{1}{16}(a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)) \qquad (121)$$

Extension 1

The sum of the area squares of a tetrahedron $\text{T}P_1P_2P_3P_4$ is given by

$$(area\triangle P_1P_2P_3)^2+(area\triangle P_1P_2P_4)^2+(area\triangle P_2P_3P_4)^2+(area\triangle P_3P_1P_4)^2$$
$$=-\frac{1}{8}(d_{12}^4+d_{13}^4+d_{14}^4+d_{23}^4+d_{24}^4+d_{34}^4-d_{12}^2d_{13}^2-d_{12}^2d_{14}^2-d_{12}^2d_{23}^2-$$
$$d_{12}^2d_{24}^2-d_{13}^2d_{14}^2-d_{13}^2d_{23}^2-d_{13}^2d_{34}^2-d_{14}^2d_{24}^2-d_{14}^2d_{34}^2-d_{23}^2d_{24}^2-$$
$$d_{23}^2d_{34}^2-d_{24}^2d_{34}^2);\ d_{ij}=\overline{P_iP_j} \qquad (122)$$

Find a classical determinant representation and an equ. (122) like *bdc representation of this string expression.

Possible solutions:

$$(area\triangle P_1P_2P_3)^2+(area\triangle P_1P_2P_4)^2+(area\triangle P_2P_3P_4)^2+(area\triangle P_3P_1P_4)^2=$$

$\frac{1}{?}$

0	d_{12}^2	d_{13}^2	d_{14}^2	d_{23}^2	d_{34}^2	d_{24}^2
d_{12}^2	0					
d_{13}^2		0				
d_{14}^2			0			
d_{23}^2				0		
d_{34}^2					0	
d_{24}^2						0

=

(All empty cells should contain 0 or 1.)

$$-\frac{1}{8}\left|\begin{array}{cccccccc} d_{12}^2 & d_{23}^2 & d_{13}^2 & d_{34}^2 & d_{14}^2 & d_{24}^2 & d_{12}^2 & \\ d_{24}^2 & d_{12}^2 & d_{23}^2 & d_{13}^2 & d_{34}^2 & d_{14}^2 & d_{24}^2 & \end{array}\right| \begin{array}{l} \rightarrow\ +\ * \\ \longrightarrow\ -\ * \end{array}$$

start stop

$$=-\frac{1}{8}(+d_{12}^2*d_{12}^2-d_{12}^2*d_{24}^2-d_{24}^2*d_{23}^2+d_{23}^2*d_{23}^2-\cdots) \qquad (123)$$

Find more applications for the *bdc notation, f. e.

• for the area of a quadrangle $\square P_1P_2P_3P_4$

$$(area\square P_1P_2P_3P_4)^2=\frac{1}{16}(4d_{13}^2d_{24}^2-(d_{12}^2-d_{23}^2+d_{34}^2-d_{14}^2)^2); \qquad (124)$$
$$d_{ij}=\overline{P_iP_j}$$

• for the volume of a tetrahedron

$$(volume\ TP_1P_2P_3P_4)^2 = \frac{1}{288}\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix} \; ; \; d_{ij} = \overline{P_iP_j} \qquad (125)$$

References:

- http://mathworld.wolfram.com/Quadrilateral.html
- http://mathworld.wolfram.com/Tetrahedron.html
- http://mathworld.wolfram.com/Cayley-MengerDeterminant.html

-

1.6 Concluding remarks

1.6.1

How about applications in the fields of Mechanical Engineering, Electrical Engineering and Physics?

1.6.2

Vektoren, Matrizen und Determinanten sind zellular aufgebaute mathematische 2D Objekte, die gestatten, komplizierte 1D (string) Ausdrücke sehr kompakt und visuell optimal zu notieren. Dieser Vorteil wird durch entsprechende Rechenregeln noch verstärkt, sie sind optimale 2D Tools für Anwendungen in Naturwissenschaften und Engineering.

Dabei sind aber nur sehr einfache Verknüpfungsmuster (Skalar- und Vektorprodukt von Vektoren, Zeile mal Spalte, Sarrus-Regel) benutzt, der Kanon der möglichen „Webmuster (weaving patterns)“ bei weitem nicht ausgeschöpft, hier schlummern noch gewaltige Reserven: go to weaving, tilings and brickwork.

Entsprechende 3D Tools (für die schon jetzt viele Anwendungen (z.B. elektrische 3D Netzwerke) absehbar sind) existieren noch nicht! Menschliche Notierungs- und Modellierungsstrukturen sind sehr stark 2D geprägt: Schrift und Schreiben, Graphik und Malerei, mathematische, technische und künstlerische Notationssysteme.

In der Natur aber sind 3D Konstruktionen die Regel:
Kristalle, Viren, Bäume, neuronale Netze (= spezielle „Rechen-strukturen“). A bionic encouragement: go to 3D. But simpler:
3D extension of the existing matrix and determinant calculus!
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Example

2D representation of the 1D (string) formula expressions for the area of a 2D triangle in function of

- the coordinates of its vertices: equs (1-4, 7, 8, 119),
- its side length: equs (120,121).

Supplement 1 to IntroAlgEngLect 1

A Comparative bdc representations of the coordinates of centroid, circumcenter, incenter and orthocenter of a triangle $\triangle P_1P_2P_3$ in function of its vertex coordinates (x_i, y_i); i=1(1)3; see 1.4.2.1

A1 Centroid

$$x_c = \frac{1}{3}\frac{\begin{vmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} = \frac{1}{3}(x_1 + x_2 + x_3)$$

(A1.1)

$$y_c = \frac{1}{3}\frac{\begin{vmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_3 \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} = \frac{1}{3}(y_1 + y_2 + y_3) \qquad \text{(A2.2)}$$

A2 Circumcenter

$$x_{cc} = \frac{1}{2}\frac{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 & y_1^2 & y_2^2 & y_3^2 \\ y_1 & y_2 & y_3 & y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad \text{(A2.1)}$$

$$y_{cc} = \frac{1}{2}\frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 & y_1^2 & y_2^2 & y_3^2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad \text{(A2.2)}$$

A3 Incenter
Find the solution!

A4 Orthocenter (A3.)

$$x_{oc} = -\frac{\begin{vmatrix} x_2x_3 & x_3x_1 & x_1x_2 & y_2y_3 & y_3y_1 & y_1y_2 \\ y_1 & y_2 & y_3 & y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad \text{(A4.1)}$$

$$y_{oc} = -\frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_1 & x_2 & x_3 \\ x_2x_3 & x_3x_1 & x_1x_2 & y_2y_3 & y_3y_1 & y_1y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} \qquad \text{(A4.2)}$$

Find a common bdc representation for the equs (A1.1, 2) – (A4.1, 2).

B Moment of inertia of the area of the simple 2D polygon $P_1P_2P_3\cdots P_n$ with respect to the

$$\begin{cases} x-axis \\ y-axis \end{cases} \rightarrow \begin{matrix} I_x(polygon\ P_1P_2P_3\cdots P_n) \\ I_y(polygon\ P_1P_2P_3\cdots P_n) \end{matrix}$$ **; see 1.5.2**

Application fields:
- Mechanics, statics, esp. calculation of the section modulus of a structural element of machines, buildings, etc;
- Pattern recognition.

$$I_x(\text{polygon } P_1P_2P_3\cdots P_n)=\oint_{\substack{\text{polygon}\\P_1P_2P_3\cdots P_n}} y(x)^2\,dxdy=\frac{1}{3}\oint_{\substack{\text{polygon}\\P_1P_2P_3\cdots P_n}} y(x)^3\,dx$$

$$=\frac{1}{3}\left\{\int_{x_1}^{x_n}\left(y_1+\frac{y_n-y_1}{x_n-x_1}(x-x_1)\right)^3dx+\cdots+\int_{x_2}^{x_1}\left(y_2+\frac{y_1-y_2}{x_1-x_2}(x-x_2)\right)^3dx\right\} \tag{B.1}$$

Substitution:

$$z=y_1+\frac{y_n-y_1}{x_n-x_1}(x-x_1) \tag{B.2}$$

$$\frac{dz}{dx}=\frac{y_n-y_1}{x_n-x_1}\rightarrow dx=\frac{x_n-x_1}{y_n-y_1}dz \tag{B.3}$$

$$x=x_1\rightarrow z=y_1\ ;\ x=x_n\rightarrow z=y_n \tag{B.4}$$

(B.2), (B.3), (B.4) in (B.1) $\rightarrow$

$$I_x(\text{polygon } P_1P_2P_3\cdots P_n)=\frac{1}{3}\left\{\frac{x_n-x_1}{y_n-y_1}\int_{y_1}^{y_n}z^3dz+\cdots+\frac{x_1-x_2}{y_1-y_2}\int_{y_2}^{y_1}z^3dz\right\}$$

$$=\frac{1}{12}\left\{(x_n-x_1)\frac{y_n^4-y_1^4}{y_n-y_1}+\cdots+(x_1-x_2)\frac{y_1^4-y_2^4}{y_1-y_2}\right\}$$

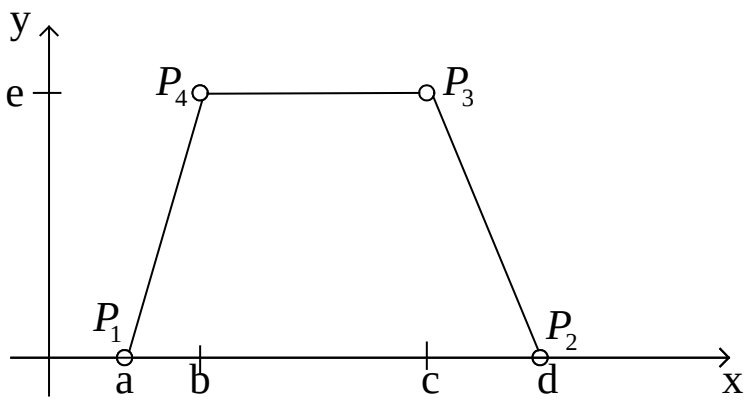
$$=\frac{1}{12}\begin{vmatrix} x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_3 & x_3 & x_3 & & x_n & x_n & x_n\\ y_1 & y_1 & y_1 & y_2 & y_2 & y_2 & y_3 & y_3 & y_3 & --- & y_n & y_n & y_n\\ y_1 & y_1 & y_1 & y_2 & y_2 & y_2 & y_3 & y_3 & y_3 & & y_n & y_n & y_n\\ y_1 & y_1 & y_1 & y_2 & y_2 & y_2 & y_3 & y_3 & y_3 & & y_n & y_n & y_n \end{vmatrix} \tag{B.5}$$

In the same way we find

$$I_y(\text{polygon } P_1P_2P_3\cdots P_n)=$$

$$=\frac{1}{12}\begin{vmatrix} x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_3 & x_3 & x_3 & & x_n & x_n & x_n\\ x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_3 & x_3 & x_3 & --- & x_n & x_n & x_n\\ x_1 & x_1 & x_1 & x_2 & x_2 & x_2 & x_3 & x_3 & x_3 & & x_n & x_n & x_n\\ y_1 & y_1 & y_1 & y_2 & y_2 & y_2 & y_3 & y_3 & y_3 & & y_n & y_n & y_n \end{vmatrix} \tag{B.6}$$

Example 1



$$P_1 = (a,0)$$
$$P_2 = (d,0)$$
$$P_3 = (c,e)$$
$$P_4 = (b,e)$$

$$I_x(\text{polygon } P_1P_2P_3\cdots P_n)$$

$$= \frac{1}{12}\begin{vmatrix} a & a & a & d & d & d & c & c & c & b & b & b \\ 0 & 0 & 0 & 0 & 0 & 0 & e & e & e & e & e & e \\ 0 & 0 & 0 & 0 & 0 & 0 & e & e & e & e & e & e \\ 0 & 0 & 0 & 0 & 0 & 0 & e & e & e & e & e & e \end{vmatrix}$$

$$= \frac{1}{12}(de^3 + 3ce^3 - ae^3 - 3be^3)$$

$$= \underline{\underline{\frac{1}{12}\big((d-a)+3(c-b)\big)e^3}} \qquad \text{(B.7)}$$

$$I_y(\text{polygon } P_1P_2P_3\cdots P_n) =$$

$$= \frac{1}{12}\begin{vmatrix} a & a & a & d & d & d & c & c & c & b & b & b \\ a & a & a & d & d & d & c & c & c & b & b & b \\ a & a & a & d & d & d & c & c & c & b & b & b \\ 0 & 0 & 0 & 0 & 0 & 0 & e & e & e & e & e & e \end{vmatrix}$$

$$= \frac{1}{12}\left((c^3 + c^2 d + cd^2 + d^3) - (a^3 + a^2 b + ab^2 + b^3)\right)e$$

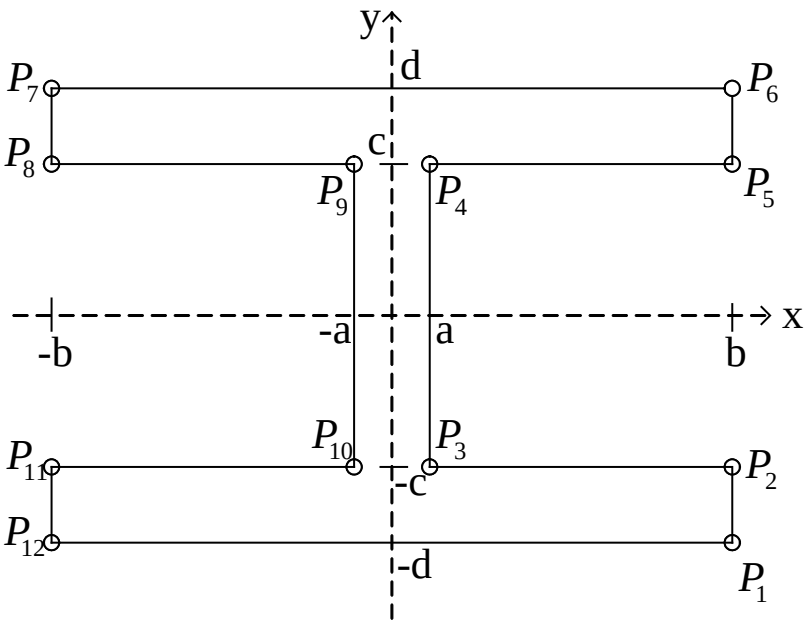
$$= \underline{\underline{\frac{1}{12}\left(\frac{d^4 - c^4}{d - c} - \frac{b^4 - a^4}{b - a}\right)e}} \qquad \text{(B.8)}$$

Comment:

$Equs \begin{cases} (B.5)\ and\ (B.7) \\ (B.6)\ and\ (B.8) \end{cases}$ show: $\begin{cases} I_x(P_1P_2P_3\cdots P_n) \\ I_y(P_1P_2P_3\cdots P_n) \end{cases}$ is invariant

regarding translation in $\begin{cases} x-direction \\ y-direction \end{cases}$.

Example 2:



$P_1 = (b, -d)$
$P_2 = (b, -c)$
$P_3 = (a, -c)$
$P_4 = (a, c)$
$P_5 = (b, c)$
$P_6 = (b, d)$
$P_7 = (-b, d)$
$P_8 = (-b, c)$
$P_9 = (-a, c)$
$P_{10} = (-a, -c)$
$P_{11} = (-b, -c)$
$P_{12} = (-b, -d)$

$I_x(\text{polygon } P_1P_2P_3\cdots P_{12}) =$

$$\frac{1}{12}\begin{vmatrix} b & b & b & b & b & b & a & a & a & a & a & a \\ -d & -d & -d & -c & -c & -c & -c & -c & -c & c & c & c \\ -d & -d & -d & -c & -c & -c & -c & -c & -c & c & c & c \\ -d & -d & -d & -c & -c & -c & -c & -c & -c & c & c & c \end{vmatrix}$$

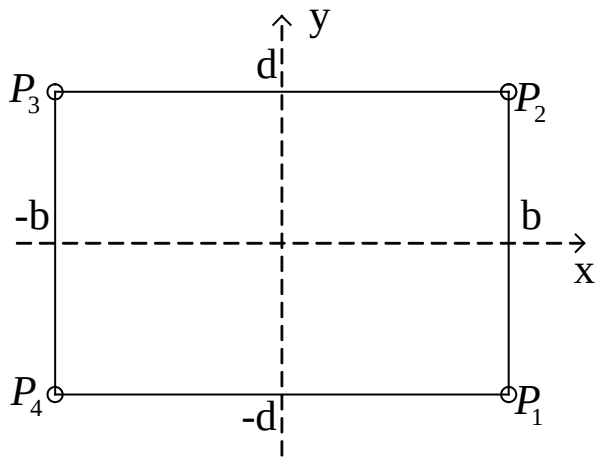
b	b	b	b	b	b	$-b$	$-b$	$-b$	$-b$	$-b$	$-b$
c	c	c	d	d	d	d	d	d	c	c	c
c	c	c	d	d	d	d	d	d	c	c	c
c	c	c	d	d	d	d	d	d	c	c	c

$-a$	$-a$	$-a$	$-a$	$-a$	$-a$	$-b$	$-b$	$-b$	$-b$	$-b$	$-b$	b	b	b
c	c	c	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-d$	$-d$	$-d$	$-d$	$-d$	$-d$
c	c	c	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-d$	$-d$	$-d$	$-d$	$-d$	$-d$
c	c	c	$-c$	$-c$	$-c$	$-c$	$-c$	$-c$	$-d$	$-d$	$-d$	$-d$	$-d$	$-d$

$$= \frac{1}{12} \cdot 16\left(ac^3 - bc^3 + bd^3\right) = \frac{4}{3}\left(ac^3 + b\left(d^3 - c^3\right)\right) \qquad \text{(B.9)}$$

$$I_y(\text{polygon } P_1P_2P_3\cdots P_{12}) = ? \qquad \text{(B.10)}$$

Example 3



$$P_1 = (b, -d)$$
$$P_2 = (b, d)$$
$$P_3 = (-b, d)$$
$$P_4 = (-b, -d)$$

$$I_x = \frac{1}{12}$$

b	b	b	b	b	b	$-b$	$-b$	$-b$	$-b$	$-b$	$-b$	b	b	b
$-d$	$-d$	$-d$	d	d	d	d	d	d	$-d$	$-d$	$-d$	$-d$	$-d$	$-d$
$-d$	$-d$	$-d$	d	d	d	d	d	d	$-d$	$-d$	$-d$	$-d$	$-d$	$-d$
$-d$	$-d$	$-d$	d	d	d	d	d	d	$-d$	$-d$	$-d$	$-d$	$-d$	$-d$

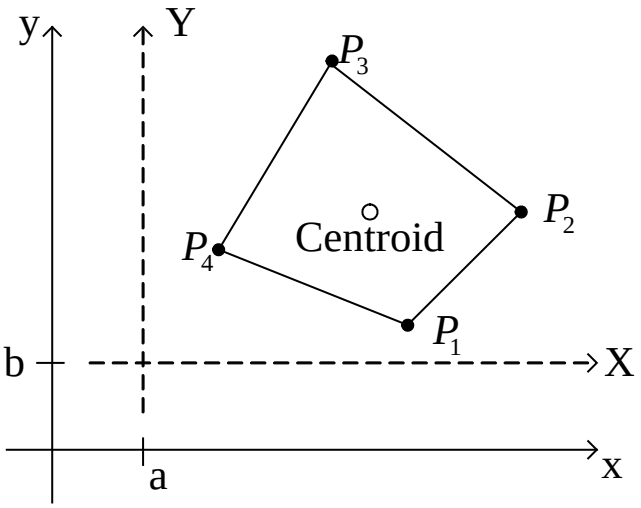
$$\underline{\underline{I_x = \frac{1}{12} \cdot 16bd^3 = \frac{4}{3}bd^3}} \qquad \text{(B.11)}$$

$$I_y = \frac{4}{3} b^3 d \qquad \text{(B.12)}$$

References:

- R. C. Hibbeler. Mechanics of Materials.
  Prentice Hall, Englewood Cliffs, 2000
- Deng Yong, Shi Wenkang, Du Feng, Liu Qi.
  A new similarity measure of generalized recognition fuzzy numbers and its application to pattern recognition
  Pattern Recognition Letters vol. 25, issue 8, pp. 875-883
  (June 2004)
- Yong Deng, Zhu Zhenfu, Liu Qi. Ranking Fuzzy Numbers with an Area Method using Radius of Gyration. r_x=sqr(I_x*area)
  Computers and Mathematics with Applications 51 (2006), 1127-1136

C Generalized parallel axis theorem of the moment of inertia of the area of a simple 2D polygon $P_1P_2P_3\cdots P_n$



x=a+X
y=b+Y

$$I_x = \frac{1}{12}\begin{vmatrix} x_1 & x_1 & x_1 & & x_n & x_n & x_n \\ y_1 & y_1 & y_1 & & y_n & y_n & y_n \\ y_1 & y_1 & y_1 & --- & y_n & y_n & y_n \\ y_1 & y_1 & y_1 & & y_n & y_n & y_n \end{vmatrix} =$$

The moment of inertia of the area of the polygon $P_1P_2P_3\cdots P_n$ with respect to the

x-axis is given by the sum

$$\left[I_X = \frac{1}{12} \begin{vmatrix} X_1 & X_1 & X_1 & & X_n & X_n & X_n \\ Y_1 & Y_1 & Y_1 & & Y_n & Y_n & Y_n \\ Y_1 & Y_1 & Y_1 & --- & Y_n & Y_n & Y_n \\ Y_1 & Y_1 & Y_1 & & Y_n & Y_n & Y_n \end{vmatrix} + \right.$$

moment of inertia of the area of the polygon $P_1P_2P_3\cdots P_n$ with respect to the X-axis

$$\left. b\cdot\frac{1}{3}\begin{vmatrix} X_1 & X_1 & & X_n & X_n \\ Y_1 & Y_1 & --- & Y_n & Y_n \\ Y_1 & Y_1 & & Y_n & Y_n \end{vmatrix} + b^2\cdot\left(area(P_1\cdots P_n) = \frac{1}{2}\begin{vmatrix} X_1 & & X_n \\ Y_1 & --- & Y_n \end{vmatrix} \right) \right.$$

(C.1)

I_x is not depending on a.

$$I_y = \frac{1}{12}\begin{vmatrix} x_1 & x_1 & x_1 & & x_n & x_n & x_n \\ x_1 & x_1 & x_1 & & x_n & x_n & x_n \\ x_1 & x_1 & x_1 & --- & x_n & x_n & x_n \\ y_1 & y_1 & y_1 & & y_n & y_n & y_n \end{vmatrix}$$

The moment of inertia of the area of the polygon $P_1P_2P_3\cdots P_n$ with respect to the y-axis is given by the sum

$$I_Y = \frac{1}{12}\begin{vmatrix} X_1 & X_1 & X_1 & & X_n & X_n & X_n \\ X_1 & X_1 & X_1 & \cdots & X_n & X_n & X_n \\ X_1 & X_1 & X_1 & & X_n & X_n & X_n \\ Y_1 & Y_1 & Y_1 & & Y_n & Y_n & Y_n \end{vmatrix} +$$

moment of inertia of the area of the polygon $P_1P_2P_3\cdots P_n$ with respect to the Y-axis

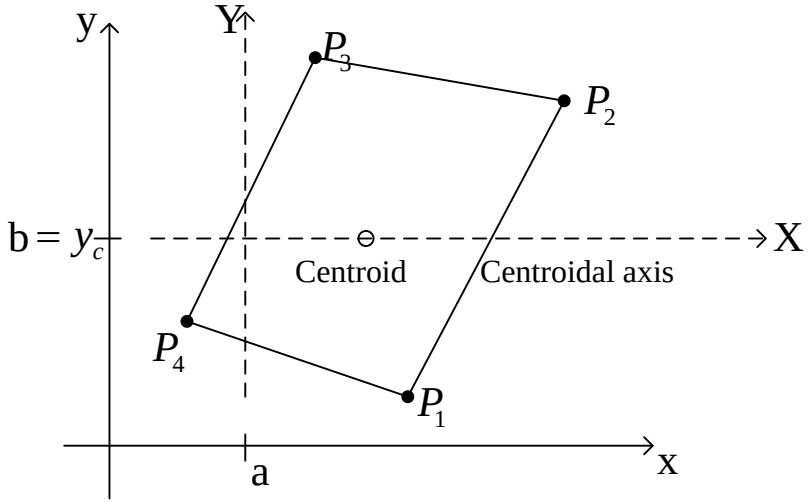
$$a\cdot\frac{1}{3}\begin{vmatrix} X_1 & X_1 & & X_n & X_n \\ X_1 & X_1 & \cdots & X_n & X_n \\ Y_1 & Y_1 & & Y_n & Y_n \end{vmatrix} + a^2\cdot\left(area(P_1\cdots P_n) = \frac{1}{2}\begin{vmatrix} X_1 & & X_n \\ Y_1 & \cdots & Y_n \end{vmatrix}\right)$$

(C.2)

I_y is not depending on b.

Special cases: parallel axis theorem

C1 X-axis is centroidal axis



If X-axis = centroidal axis then

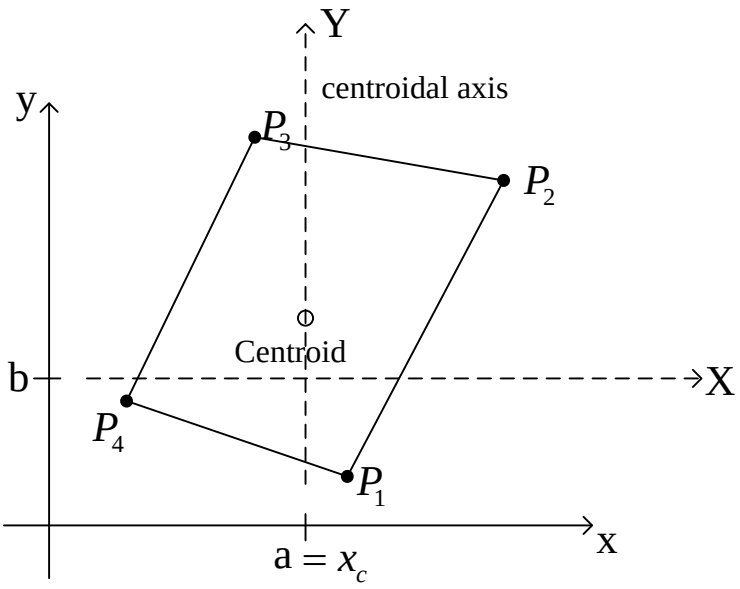
$$\begin{vmatrix} X_1 & X_1 & & X_n & X_n \\ Y_1 & Y_1 & \cdots & Y_n & Y_n \\ Y_1 & Y_1 & & Y_n & Y_n \end{vmatrix} = 0$$

(C1.1)

and

$$I_x = I_X + b^2 \cdot area(P_1P_2P_3\cdots P_n) \;\; ; \;\; b = y_c \qquad \text{(C1.2)}$$

C2 Y-axis is centroidal axis



If Y-axis = centroidal axis then

$$\begin{vmatrix} X_1 & X_1 & & X_n & X_n \\ X_1 & X_1 & --- & X_n & X_n \\ Y_1 & Y_1 & & Y_n & Y_n \end{vmatrix} = 0 \qquad \text{(C2.1)}$$

and

$$I_y = I_Y + a^2 \cdot area(P_1P_2P_3\cdots P_n) \;\; ; \;\; a = x_c \qquad \text{(C2.2)}$$

Supplement 2 to IntroAlgEngLect 1

The output sequence

$$\langle y(t)\rangle_{t=0}^{\infty} = y(0), y(1), y(2), y(3), \cdots \;\; ; y \in \mathbb{R} \qquad (175)$$

of a numerical algorithm has the following features

- y(t) converges to s

$$\lim_{t\to\infty} y(t) = s \;\; ; \;\; y(t) \neq s \text{ for all finite t} \qquad (176)$$

- y(t) converges linearly, that means:
  the difference quotient of y(t)

$$dy(t) = \frac{y(t) - y(t+1)}{y(t+1) - y(t+2)} \ , \tag{177}$$

  - a measure for the convergence intensity and order -
  converges to d

$$\lim_{t\to\infty} dy(t) = d \ . \tag{178}$$

Equ. (178) means also: the larger t the more exact is the approximation

$$dy(t) \approx dy(t+1) \ . \tag{179}$$

Equs (179) and (177) give with

$$y(t)=y,\ y(t+1)=y',\ y(t+2)=y'',\ y(t+3)=y''' \tag{180}$$

in bcd notation another condition for linear convergence

$$\lim_{t\to\infty} \begin{vmatrix} y & y' & y'' \\ y' & y'' & y''' \end{vmatrix} = 0 \ , \tag{181}$$

that means: the point $P_1 = (y, y')$, $P_2 = (y', y'')$ and $P_3 = (y'', y''')$ are collinear for $t \to \infty$.

Questions

- What is matter with the 2-D triangle $P_1P_2P_3$ for finite t, esp. is (see equ. (71))

$$\tan\alpha_1 = \frac{\begin{vmatrix} y & y' & y'' \\ y' & y'' & y''' \end{vmatrix}}{\begin{vmatrix} y & y' & y' & y'' \\ y''' & y & y' & y'' \end{vmatrix}} \tag{182}$$

  a useful tool to evaluate the convergence process of $\langle y(t) \rangle_{t=0}^{\infty}$?

- A condition for quadratic convergence of the series equ. (175) with equ. (176) is

$$\lim_{t\to\infty} (dy(t+1) - (dy(t))^2) = 0 \ . \tag{183}$$

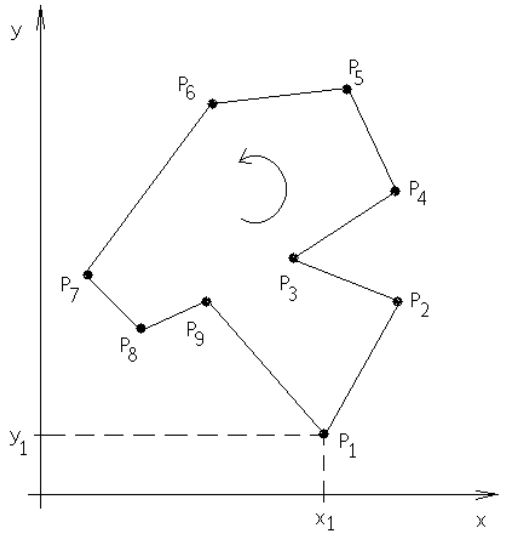
  Equ. (183) gives with equs (177) and (180) the condition

$$\lim_{t\to\infty} (yyy''+yy'y'''+yy'y'''+y'y'y''+y'y'y''+y'y'y''+y'y'y'' + y'y''y''+yyy'''+yy'y''+yy'y''+y'y'y'+y'y'y'''+y'y''y'' + y'y'y''+y'y''y'') = 0 \qquad (184)$$

Find a "convincing" bcd or classical matrix or other ……. representation of equ. (184).

Supplement 3 to IntroAlgEngLect 1

1.5.2 Advanced exercise 2
A bridge building to calculus.



The contour integral around a simple 2D polygon $\square P_1P_2P_3P_4P_5P_6\cdots P_n$

$$I(\mathrm{k},\mathrm{l},\mathrm{n}) = \oint_{\substack{\text{polygon}\\ P_1P_2P_3\cdots P_n}} \mathrm{x}^{\mathrm{k}}\, y(x)^{l}\, dx \qquad (98.1)$$

given by the coordinates of its vertices

$$P_i = (x_i, y_i)\ ;\ \mathrm{i}=1(1)\mathrm{n} \qquad (98.2)$$

can be represented by a bdc.

and

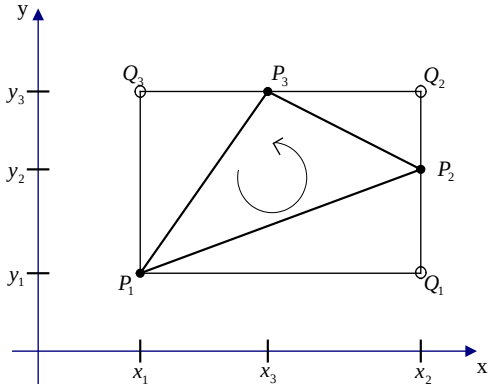
M. Tajmar, F. Plesescu, B. Seifert, K. Marhold.
Measurement of Gravitometric and Acceleration Fields Around Rotating Superconductors.
http://www.arxiv.org/ftp/gr-qc/papers/0610/0610015.pdf

= ?

Supplement 4 to IntroAlgEngLect 1
Transform the string formula representation
$(1\text{-}z)^2 - kz^2 - z^2(1\text{-}z)^2((1+k)z - k) = 0$
for the calculation of the Lagrange point L1 between Earth and Moon into a "convincing" 2D bdc representation.
z = distance Earth – L1/distance Earth – Moon ≈ 0.849..
k = mass of Moon/mass of Earth ≈ 0.01229..

- http://www.answers.com/Lagrangian%20point
- http://www.ottisoft.com/samplact/Lagrange%20poinf%20L1.htm
- http://map.gsfc.nasa.gov/ContentMedia/lagrange.pdf

Supplement 5 to IntroAlgEngLect 1
Variations on the theme



Given a planar triangle $\triangle P_1P_2P_3$ by the coordinates of its vertices $P_i = (x_i, y_i)$; $i = 1(1)3$, **the simplest interaction between the points *P₁*, *P₂* and *P₃*.**

Find a formula expression for the area of $\triangle P_1P_2P_3$ in function of the coordinates x_i and y_i taking a simplest and shortest way.

The aforementioned semantics of the 2D 3point arrangement as the geometric object triangle (with area, sum of side squares, angles,

special points, etc) provocates immediately the question for other interpretations, f.e.

* three sphere condensator (see IntroAlgEng6.4.2)
* three input automaton
* three phase magnetic rotating field, three phase induction motor, three phase reluctance motor
* 3-body problem (mass center, potential, moments, Newtonian 3-body problem),
* triple synchronization and resonance effects:
  - 3 mechanical coupled (f.e. on a suitable common metal baseplate) mechanical but also electronic clockworks
  - 3 electrical or magnetic coupled systems (oscillation circuits, motors, etc.)
  - 3 acoustic coupled biological systems
* local triple operators for transitivity and metrics
  - $(P_1 rel P_2)\&(P_2 rel P_3) \rightarrow (P_1 rel P_3)$
  - $dist(P_1,P_2)+dist(P_2,P_3) \geq dist(P_1,P_3)$

References

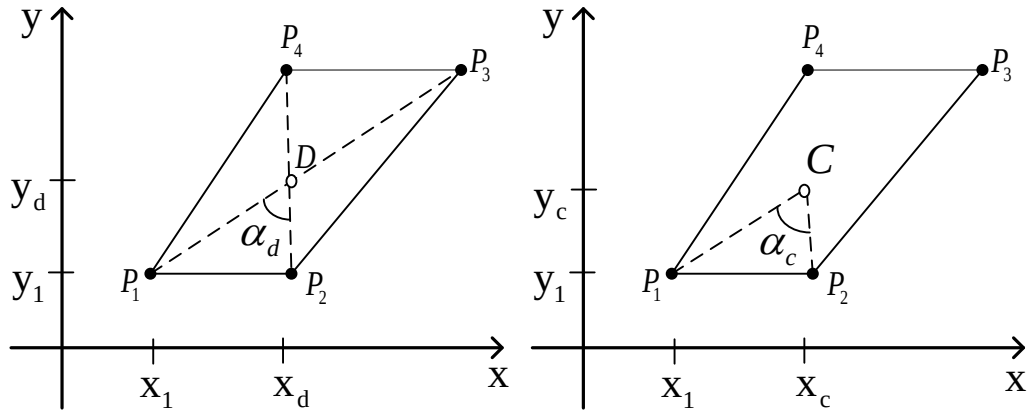
- 2D homogenous sphere capacitor net metamaterial.
  E. Shamonia, L. Solymar. Metamaterials: How the subject started.
  Metamaterials **1** (2007), pp.12-18
  See also IntroAlgEng1Appendix2#2
- n-body problem
  http://www.answers.com/n-boy%20problem
- synchronization and resonance effects
  http://www.timezone.com/library/rdnotebook/200503149532
  R. DeCorte. Jaeger-LeCoultre Gyrotourbillon.
  Of great interest is in this context the scientific work of S.E. Shnoll about the synchronizing influence of the “global cosmic time wave”.
  S.E. Shnoll et al. Realization of discrete states during fluctuations in macroscopic processes.
  Physics Uspekhi 41 (10), p. 1026, 1998
  www.ufn.

Start up 1 to IntroAlgEngLect 1

If you find however, wherever such a formula expression, regard it as a present of the gods, try immediately to extend, to mutate it in all possible directions!

Die Spur ist ausgelegt. Die Inszenierung kann beginnen.
The trail is marked. The production of the play can start.

Example 1
Analytic geometry in 2D.
1.1 Diagonal intersection point D and centroid C of a 2D simple convex quadrangle.



$$z_d = x_d + jy_d = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \; ; \quad z_i = x_i + jy_i \qquad \text{(E1. 1)}$$

$$z_c = x_c + jy_c = \frac{1}{3} \cdot \frac{\begin{vmatrix} x_1 & x_1 & x_2 & x_2 & x_3 & x_3 & x_4 & x_4 \\ z_1 & z_1 & z_2 & z_2 & z_3 & z_3 & z_4 & z_4 \\ y_1 & y_1 & y_2 & y_2 & y_3 & y_3 & y_4 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \qquad \text{(E1. 2)}$$

duplication effect!

$$z_d + 3z_c = z_1 + z_2 + z_3 + z_4 \qquad \text{(E1. 3)}$$

In general is $D \neq C,\ z_d \neq z_c$ (E1. 4)

$$D = C \ \text{ iff } \ (z_1 + z_2 + z_3 + z_4)\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = 4\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix},$$

$$\text{then we have } z_d = z_c = \frac{1}{4}(z_1 + z_2 + z_3 + z_4). \qquad \text{(E1. 5)}$$

1.2 Areas of the D and C based sectors of this triangle

$$\text{area}(\triangle P_1P_2D) = \frac{\text{area}(\triangle P_1P_2P_3)\cdot \text{area}(\triangle P_1P_2P_4)}{\text{area}(\square P_1P_2P_3P_4)}$$

$$= \frac{1}{2}\frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}\cdot\begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \qquad \text{(E1. 6)}$$

$$\text{area}(\triangle P_1P_2D) + 3\,\text{area}(\triangle P_1P_2C) = \text{area}(\triangle P_1P_2P_3) + \text{area}(\triangle P_1P_2P_4)$$

$$= \frac{1}{2}\left(\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \end{vmatrix}\right)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 & x_2 & x_4 \\ y_1 & y_2 & y_3 & y_1 & y_2 & y_4 \end{vmatrix} \qquad \text{(E1. 7)}$$

duplication effect!

Basis for equ. (E1.7) is the bcd addition theorem:

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n \end{vmatrix} \pm \begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_{n+1} \\ y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_{n+1} \end{vmatrix} =$$

$$\begin{matrix} + \\ \\ - \end{matrix}\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n & x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_{n+1} \\ y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n & y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_{n+1} \end{vmatrix}$$

$$\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\ y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n \end{vmatrix}$$

(E1. 8)

Find other bcd transformation rules.

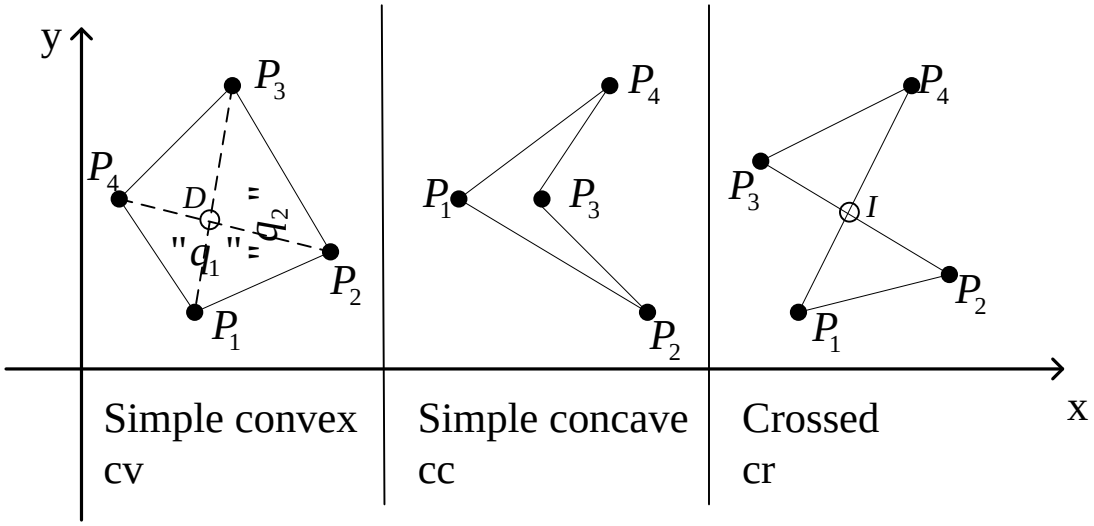
1.3 Central angles of the D and C based sectors of the quadrangle 1.1

$$\tan\alpha_d = \frac{\begin{vmatrix} x_1 & x_2 & x_d \\ y_1 & y_2 & y_d \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_d & y_d \\ y_d & x_2 & y_2 & x_d \end{vmatrix}} = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix}} \qquad \text{(E1. 9)}$$

$$\tan\alpha_c = \frac{\begin{vmatrix} x_1 & x_2 & x_c \\ y_1 & y_2 & y_c \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_c & y_c \\ y_c & x_2 & y_2 & x_c \end{vmatrix}} = ? \qquad \text{(E1. 10)}$$

Find a similar expression like (E1.9).

1.4 Classification of 2D quadrilaterals



With the “discriminators”.

$$q_1 = \frac{2\,area_{\triangle}P_1P_2P_4}{2\,area_{\square}P_1P_2P_3P_4} = \frac{\begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \qquad \text{(E1.11)}$$

$$q_1 = \frac{2\,area_{\triangle}P_1P_2P_3}{2\,area_{\square}P_1P_2P_3P_4} = \frac{\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \qquad \text{(E1.12)}$$

and the quantification encoding:

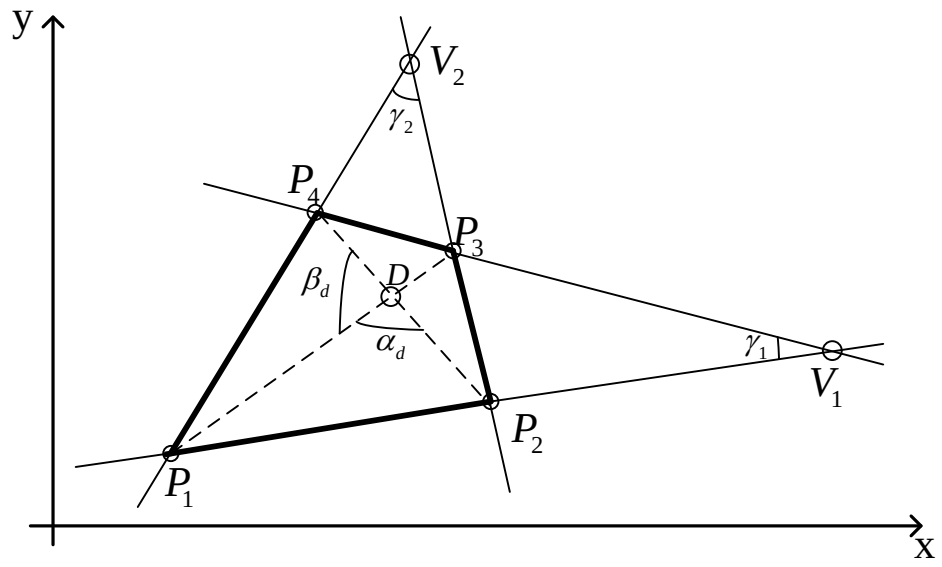
$$Q_i = \begin{cases} 0 & q_i < 0 \\ 1 \quad \text{if} & 0 < q_i < 1 \\ 2 & 1 < q_i \end{cases} \qquad \text{(E1.13)}$$

we get a very simple 2D quadrangle classification table with an extensively inherent symmetry.

If		then $area_{\square}P_1P_2P_3P_4$ is:	
Q_1	Q_2		
1	1	convex (cv)	
0	1	concave (cc), the concave corner is located at point	P_1
1	0		P_2
2	1		P_3
1	2		P_4
0	2	crossed (cr), the crossing sides are: the intersection point I is	$\overline{P_1P_2}$ and $\overline{P_3P_4}$; $\square P_1P_2P_3P_4$
2	0		
0	0		$\overline{P_1P_4}$ and $\overline{P_2P_3}$; $\square P_1P_2P_3P_4$

2	2	the diagonal point D in the quadrangle	

1.5 Diagonal point D and vanishing points V_1 and V_2 of a cv quadrangle



$$z_d = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}} \qquad \text{(E1.13)}$$

$$z_{V_1} = \frac{\begin{vmatrix} x_4 & x_1 & x_3 & x_2 \\ z_4 & z_1 & z_3 & z_2 \\ y_4 & y_1 & y_3 & y_2 \end{vmatrix}}{\begin{vmatrix} x_4 & x_1 & x_3 & x_2 \\ y_4 & y_1 & y_3 & y_2 \end{vmatrix}} \qquad \text{(E1.14)}$$

$$z_{V_2} = \frac{\begin{vmatrix} x_1 & x_2 & x_4 & x_3 \\ z_1 & z_2 & z_4 & z_3 \\ y_1 & y_2 & y_4 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 & x_4 & x_3 \end{vmatrix}}$$

(E1.15)

1.6 Diagonal angles α_d and β_d and vanishing angles γ_1 and γ_2 of a cv quadrangle

$$\tan\alpha_d = \frac{\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_3 & y_3 \\ y_4 & x_2 & y_2 & x_4 \end{vmatrix}} \tag{E1.16}$$

$$\tan\gamma_2 = \frac{\begin{vmatrix} x_3 & x_4 & x_2 \\ y_3 & y_4 & y_2 \end{vmatrix}\begin{vmatrix} x_4 & y_4 & x_1 & y_1 \\ y_3 & x_4 & y_4 & x_3 \end{vmatrix} + \begin{vmatrix} x_4 & x_1 & x_3 \\ y_4 & y_1 & y_3 \end{vmatrix}\begin{vmatrix} x_3 & y_3 & x_4 & y_4 \\ y_2 & x_3 & y_3 & x_2 \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 & x_4 & y_4 \\ y_2 & x_3 & y_3 & x_2 \end{vmatrix}\begin{vmatrix} x_4 & y_4 & x_1 & y_1 \\ y_3 & x_4 & y_4 & x_3 \end{vmatrix} - \begin{vmatrix} x_3 & x_4 & x_2 \\ y_3 & y_4 & y_2 \end{vmatrix}\begin{vmatrix} x_4 & x_1 & x_3 \\ y_4 & y_1 & y_3 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} x_1 & x_2 & x_4 & x_3 \\ y_1 & y_2 & y_4 & y_3 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & x_4 & y_4 \\ y_3 & x_2 & y_2 & x_3 \end{vmatrix}} \tag{E1.17}$$

$$\tan\beta_d = \frac{\begin{vmatrix} x_4 & x_1 & x_2 & x_3 \\ y_4 & y_1 & y_2 & y_3 \end{vmatrix}}{\begin{vmatrix} x_4 & y_4 & x_2 & y_2 \\ y_3 & x_1 & y_1 & x_3 \end{vmatrix}} \tag{E1.18}$$

$$\tan\gamma_1 = \frac{\begin{vmatrix} x_4 & x_1 & x_3 & x_2 \\ y_4 & y_1 & y_3 & y_2 \end{vmatrix}}{\begin{vmatrix} x_4 & y_4 & x_3 & y_3 \\ y_2 & x_1 & y_1 & x_2 \end{vmatrix}} \tag{E1.19}$$

Examination question 1

The double area of a triangle $\triangle P_1P_2P_3$ in 2D with $P_i=(x_i,y_i)$; i=1,2,3

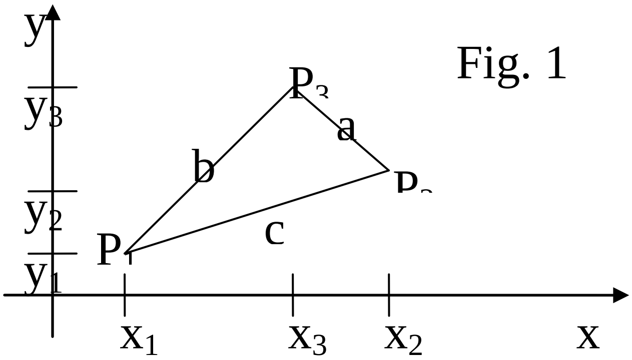


Fig. 1

is given by the unparenthesized string expression

$$2A(\Delta P_1P_2P_3) = x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3 \qquad (1)$$

You need here for calculation 6 multiplications. Transform equ. (1) by parethis with the goal to minimize the number of multiplications. Find the minimal solution and give an algebraic and geometric interpretation of your result.

Formulae manipulation skills – essential part of algorithm design! You must train these skills.

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Examination question 2

Find a symmetric, simple, impressive matrix representation of equ. (1) in exqu. 1. Is your result a “blue ocean” solution (extensible: triangle, quadrangle, n-gon;…)?

Blue ocean solutions are important for algorithm design, but also for business management, industrial design, product advertisement…
See:

- W. C. Kim; R. Mauborgne: Blue Ocean Strategy: How to Create Uncontested
  Market Space and Make Competition Irrelevant.
  Harvard Business School Publishing Corporation 2005
  ISBN 1-59139-619-0
- T. Kelley: J. Littman: The Art of Innovation: Lessons on Creativity from IDEO,
  America’s Leading Design Firm.
  Harper Collins Business 2002
  ISBN 0-00-710293-3

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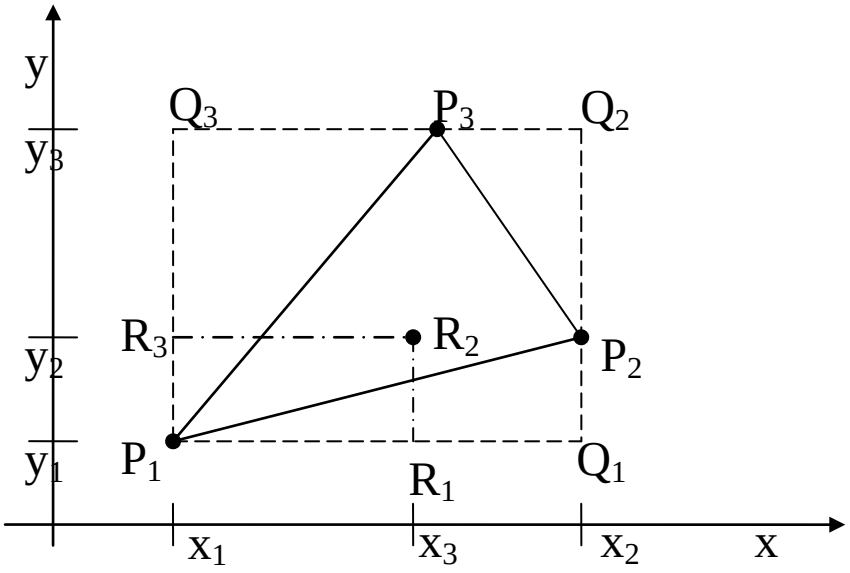
Examination question 3

Find symmetric, simple, impressive (classical) determinant and corresponding matrix representations for the area of a triangle △$P_1P_2P_3$ (Fig. 1 in exqu. 1) in function of its side lengths $\overline{P_1P_2} = c$; $\overline{P_2P_3} = a$; $\overline{P_3P_1} = b$. Which of your results are “blue ocean” solutions (extensible from triangle to quadrangle to n-gon; extensible from triangle to tetrahedron,…)?

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Examination question 1

Solution



$$A(\Delta P_1P_2P_3) = A(P_1Q_1Q_2Q_3) - A(\Delta P_1Q_1P_2) - A(P_2Q_2P_3) - A(\Delta P_3Q_3P_1)$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

$$= \frac{1}{2}(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)) \qquad 3 \text{ Mu}$$

$$= \frac{1}{2}((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)) \qquad \underline{2 \text{ Mu!}}$$

Examination question 2

Solution

a) Matrix representation of $2A(\Delta P_1P_2P_3) = x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3$

$$\begin{pmatrix} 2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \qquad (1)$$

coupling matrix $\underline{c}$

$\det \underline{c} = 0$

b) Extension to n–gon

$$\begin{pmatrix} 2A(\square P_1P_2P_3P_4 \ldots P_n) \end{pmatrix}$$

x_1	x_2	x_3	x_4	x_n

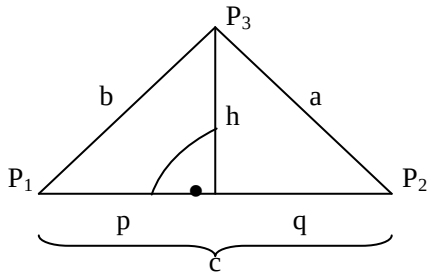
0	1	0	0	-1	y_1
-1	0	1	0	0	y_2
0	-1	0	1	0	y_3
0	0	-1	0	1	y_4
1	0	0	-1	0	y_n

(2)

Examination question 3

Solution

a) Find $A(\triangle P_1P_2P_3) = f(a, b, c)$



$2A(\triangle P_1P_2P_3) = hc \rightarrow 4A^2 = h^2c^2$

(1)

$b^2 = h^2 + p^2$

(2)

$a^2 = h^2 + q^2$

(3)

$c = p + q$

(4)

---

(2) – (3) → $b^2 - a^2 = p^2 – q^2 = (p + q)(p – q)$

(5)

Substitute (4) in (5) → $p – q = \dfrac{b^2 - a^2}{c}$

(6)

(4) + (6) → $p = \dfrac{b^2 - a^2 + c^2}{2c}$

(7)

(7) in (2) → $h^2 = b^2 - \dfrac{\left(b^2 - a^2 + c^2\right)^2}{4c^2} = \dfrac{4b^2c^2 - \left(b^2 - a^2 + c^2\right)^2}{4c^2}$

$$= \frac{1}{4c^2}\left(4b^2c^2 + 2a^2b^2 - 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4\right)$$

$$= \frac{1}{4c^2}\left(2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - \left(a^4 + b^4 + c^4\right)\right)$$

(8)

(8) in (1) → $\underline{-16A^2 = a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)}$

(9)

symmetric expression!
permutation – invariant

b) Classical determinant representations of equ. (9)

$-16A^2 =$

0	1	1	1
1	0	a^2	b^2
1	a^2	0	c^2
1	b^2	c^2	0

=

0	1	1	1
1	0	$\overline{P_1P_2}^2$	$\overline{P_1P_3}^2$
1	$\overline{P_2P_1}^2$	0	$\overline{P_2P_3}^2$
1	$\overline{P_3P_1}^2$	$\overline{P_3P_2}^2$	0

(10,1)

=

0	a^2	b^2	c^2
a^2	0	1	1
b^2	1	0	1
c^2	1	1	0

=

0	a^2	b^2	c^2
a^2	0	c^2	b^2
b^2	c^2	0	a^2
c^2	b^2	a^2	0

(12,13)

c) Matrix representations

$16A^2 =$ | a^2 | b^2 | c^2 | ×

-1	+1 (14)	+1
+1	-1	+1
+1	+1	-1

×

a^2
b^2
c^2

$16A^2 =$ | a^2 | b^2 | c^2 | ×

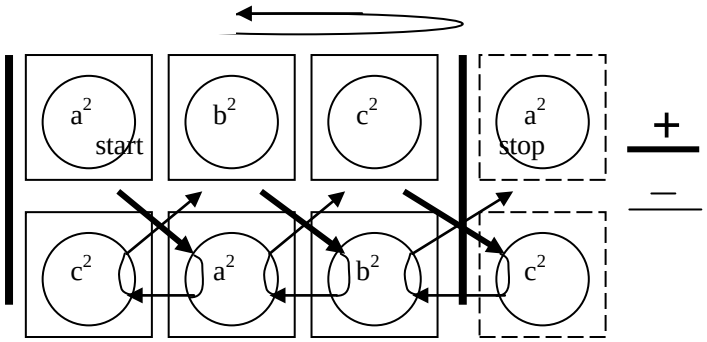
0	1	1
1	0	1
1	1	0

-1 ×

a^2
b^2
c^2

(15)

d) Shortened determinant representation



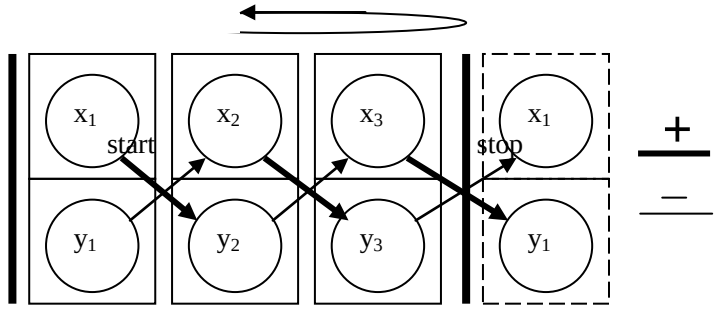
$)-16A^2 =$ (16.1

$= a^2 * a^2 - a^2 * c^2 - c^2 * b^2 + b^2 * b^2 - b^2 * c^2 - a^2 * c^2 + c^2 * c^2 - c^2 * b^2 - b^2 * a^2$

$= a^4 + b^4 + c^4 - 2(a^2b^2 + a^2c^2 + b^2c^2)$

"Array + Variables (operands) + Navigation + Operations (operators)"

Compare "Sarrus navigation":

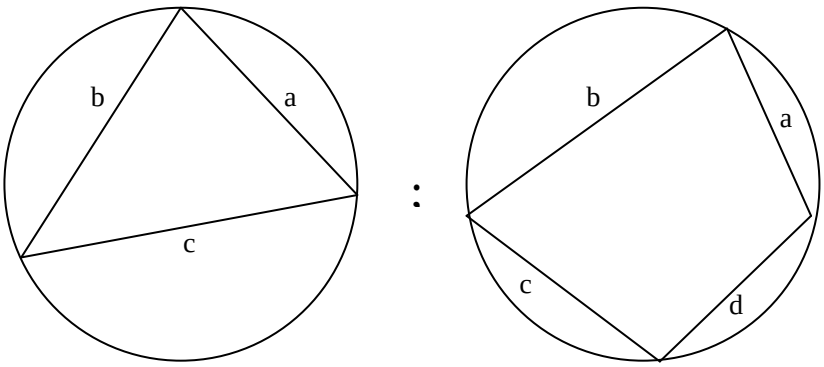


$2A^2 =$ (16.2)

$= x_1y_2 - y_1y_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1$

e) Extensions

e1) Triangle → cyclic quadrangle



$-16A^2 =$

0	a	b	c
a	0	c	b
b	c	0	a

; $-6A^2 =$

-d	a	b	c
a	-d	c	b
b	c	-d	a

(17) (18)

e2) Line segment → trangle → tetrahedron

$(-1)^0*2^1*(1!)^2*L^2 =$

j \ i	0	1	2
0	0	1	1
1	1	0	$\overline{P_1P_2}^2$
2	1	$\overline{P_2P_1}^2$	0

(19)

$(-1)^1*2^2*(2!)^2*A^2 =$

	0	1	2	3
0	0	1	1	1
1	1	0	$\overline{P_1P_2}^2$	$\overline{P_1P_3}^2$
2	1	$\overline{P_2P_1}^2$	0	$\overline{P_2P_3}^2$
3	1	$\overline{P_3P_1}^2$	$\overline{P_3P_2}^2$	0

(20)

$(-1)^2*2^3*(3!)^2*V^2 =$

	0	1	2	3	4
0	0	1	1	1	1
1	1	0	$\overline{P_1P_2}^2$	$\overline{P_1P_3}^2$	$\overline{P_1P_4}^2$
2	1	$\overline{P_2P_1}^2$	0	$\overline{P_2P_3}^2$	$\overline{P_2P_4}^2$
3	1	$\overline{P_3P_1}^2$	$\overline{P_3P_2}^2$	0	$\overline{P_3P_4}^2$
4	1	$\overline{P_4P_1}^2$	$\overline{P_4P_2}^2$	$\overline{P_4P_3}^2$	0

(21)

For more details see
- http://www.DresdenAlgorithmicsChannel.de , Modul 42
- http://mathworld.wolfram.com/Cayley-MengerDeterminant.html .