Small challenges to go, part 2

Ch 4
In this falling rain, where are you off to snail?
Wohin willst du Schnecke bei diesem Regen kriechen?
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Visual resonance

A suitable 2D representation (matrix product, determinant, permanent, …) of a 1D (string) notated formula reveals often an unexpectedly simple "inner structure". Simple means here:

** visually convincing, mental resonance, beauty
** extensible
** transferable
** synergetic, broadband merging is possible.

A collection of determinant expressions for the Bernoulli numbers

$$B_0 = 1;\ B_1 = -\frac{1}{2};\ B_2 = \frac{1}{6};\ B_4 = -\frac{1}{30};\ B_6 = \frac{1}{42};\ B_8 = -\frac{1}{30};\ \ldots$$

$$B_{2k+1} = 0;\ k = 1,2,3,\ldots$$

http://mathworld.wolfram.com/BernoulliNumber.html

$$\frac{B_n}{n!} = \begin{vmatrix} \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 & 0 \\ \frac{1}{6!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{5!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix} ; n = 2,3,4,...$$

n by n determinant (1)

$$\frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1}}{2^{2n}(2^{2n}-1)} \begin{vmatrix} \frac{1}{1!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow &] \end{vmatrix} \quad ; \quad n=1,2,3,4,..$$

n by n determinant (2)

$$\frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1}}{2^{2n}-2} \begin{vmatrix} \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & 0 & \rightarrow \\ \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & 0 & \rightarrow \\ \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \frac{1}{1!} & \rightarrow \\ \frac{1}{11!} & \frac{1}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow &] \end{vmatrix} \quad ; \quad n=1,2,3,4,..$$

n by n determinant (3)

For the same B_{2k}, the generating determinant in equ. $\begin{cases}(1)\\(3)\end{cases}$ has the size $\begin{cases}2k \text{ by } 2k\\ k \text{ by } k\end{cases}$, this represents a compression by the factor 2!

Understand the transformation equ. (1) to equ. (2) to equ. (3) step by step.

The „even counterpart“ to the determinant in equ. (3) is the Euler numbers generating determinant equ. (3^{*})

$$\frac{E_{2n}}{(2n)!} = (-1)^n \begin{vmatrix} \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & 0 & \rightarrow \\ \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & 0 & \rightarrow \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & 0 & \rightarrow \\ \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \frac{1}{0!} & \rightarrow \\ \frac{1}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & \rightarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow &] \end{vmatrix} \quad ; \quad n=1,2,3,4,..$$

n by n determinant (3^*)

The comparison of the equs (3) and (3^*) shows the close structural connection between Bernoulli and Euler numbers.

Bernoulli numbers generated by Pascal's triangle

$$
\begin{vmatrix}
1 & & & & & & & & & \\
1 & 1 & & & & & & & & \\
1 & 2 & 1 & & & & & & & \\
1 & 3 & 3 & 1 & & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & & \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & & \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1
\end{vmatrix} \quad (4)
$$

$$\begin{array}{cccccc} 1 & 1 & 0 & -4 & 0 & 120 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = (2k+1=7)!*B_{2k=6} \qquad (5)$$

$(2k=6)$ by $(2k=6)$ determinant

Understand the transformation from equ. (1) to equ. (3) to equ. (5) step by step.

To reduce the order of the B_{2k} generating determinant from 2k to k, we split the determinant equ. (6) into two parts, red and blue

$$
\begin{vmatrix}
1 & & & & & & & & & \\
1 & 1 & & & & & & & & \\
1 & 2 & 1 & & & & & & & \\
1 & 3 & 3 & 1 & & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & & \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & & \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1
\end{vmatrix} \qquad (6.1)
$$

$$
\mathrm{dr}(k=7) = \begin{vmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 6 & 1 & 0 & 0 & 0 \\
0 & 0 & 5 & 10 & 1 & 0 & 0 \\
0 & 0 & 1 & 15 & 15 & 1 & 0 \\
0 & 0 & 0 & 7 & 35 & 21 & 0 \\
0 & 0 & 0 & 1 & 28 & 70 & 28
\end{vmatrix} \qquad (6.2)
$$

$$db(k=7)=\begin{vmatrix} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 20 & 6 & 0 & 0 \\ 0 & 0 & 1 & 21 & 35 & 7 & 0 \\ 0 & 0 & 0 & 8 & 56 & 56 & 8 \\ 0 & 0 & 0 & 1 & 36 & 126 & 84 \end{vmatrix} \quad (6.3)$$

The first step to analyze the determinants equs (6.2) and (6.3) is their factorization.

Factorization of $dr(k)$

$dr(1)=1$

$dr(2)=3$

$dr(3)=17$

$dr(4)=155=5*31$

$dr(5)=2073=3*691$ Bernoulli marker!

$dr(6)=38227=7*43*127$

$dr(7)=929569=257*3617$

$dr(8)=28820619=3^2*73*43867$

Factorization of db(k)

db(1)=1

db(2)=4=2^2

db(3)=36=2^2*3^2

db(4)=600=2^3*3*5^2

db(5)=16584=2^3*3*691 Bernoulli marker!

db(6)=705600=$2^6*3^2*5^2*7^2$

db(7)=43751232=2^6*3^3*7*3617

db(8)=379010880=$2^7*3^3*5^2$*43867

We find in both cases the "Bernoulli marker" 691 as a factor. This means that Bernoulli numbers are factors of dr(k) and db(k).
By trial and error and (unproven) generalization we obtain finally

$$dr(k) = (-1)^k 2(2^{2(k+1)} - 1)B_{2(k+1)} \qquad (7.1)$$

$$db(k) = (-1)^k (2k+3)[(k+2)!]B_{2(k+1)} \qquad (7.2)$$

$$\lim_{k\to\infty}\binom{2k+4}{2}\frac{dr(k)}{dr(k+1)}=\frac{\pi^2}{2} \qquad (8.1)$$

$$\lim_{k\to\infty}\binom{2k+6}{3}\frac{db(k)}{db(k+1)}=\frac{4\pi^2}{3} \qquad (8.2)$$

In both cases we find the same related

difference quotient

$$df=\lim_{k\to\infty}df(k)=\frac{f(k)-f(k+1)}{f(k+1)-f(k+2)}=9 \qquad (9)$$

and Linear convergence acceleration

$$f1(k)=\frac{df}{df-1}f(k+1)-\frac{1}{df-1}f(k) \qquad (10)$$

For the same B_{2k}, the generating determinant in equ. $\begin{cases}(5)\\(7)\end{cases}$ has the size $\begin{cases}2k \text{ by } 2k\\ k\text{-}1 \text{ by } k\text{-}1\end{cases}$, this represents a compression by the factor 2!

**** And where is the E_{2k} voice in this**

Pascal's triangle choir? Add it!

**** π is encoded in Pascal's triangle, see Equs (8.1) and (8.2). Maybe e too?**

[1] http://mathworld.wolfram.com/BernoulliNumber.html

[2] http://mathworld.wolfram.com/EulerNumber.html

[3] http://en.wikipedia.org/wiki/691_(number)

In number theory, 691 is a "marker" (similar to the radioactive markers in biology): whenever it appears in a computation, one can be sure that Bernoulli numbers are involved.

Ch 5

A compound divisibility rule

In [1,2,3] you find rules for divisibility by individual primes, f.e.

"A number of the form $10x + y$ is divisible by 7 if and only if $x - 2y$ is divisible by 7. In other words, subtract twice the last digit from the number formed by the remaining digits. Continue to do this until a number known to be divisible by 7 is obtained. The original

number is divisible by 7 if and only if the number obtained using this procedure is divisible by 7.” [2]

“Add three times the last digit to the remaining leading truncated number. If the result is divisible by 29, then so was the first number. Apply this rule over and over again as necessary.“ [3]

Find basing on these rules compound and simple (as simple as possible!) divisibility tests like

"A natural number n=ax+by is $\begin{cases} \text{as well as} \\ \text{not} \end{cases}$ divisible by 3, 7, 11, 13, 17, and 19 if and only if ..."

Application number lock and key?

References

[1] http://de.wikipedia.org/wiki/Teilbarkeit

[2] http://en.wikipedia.org/wiki/Divisibility_rule

[3] http://www.savory.de/maths1.htm

Even more

Find a simplest possible algorithmic check:
"An odd natural number is indivisible by the first n primes."

Ch 6

An ambitious and challenging task

Eine anspruchsvolle Aufgabe

Given is the transcendent number

Gegeben ist die transzendente Zahl

ρ = 3.089548294173952229627346519086913821436997238332206901789034426123526911623810540549345148236642814177544411727855800761269876344210810880383963847675303873580913616189568171630892991830108486995416 6…

** **Decode from ρ the leading part of the prime sequence.**

Dekodieren Sie hieraus den Anfangsteil der Primzahlfolge.

** **Find an infinite expression (f.e. series, product, continued fraction, nested products, nested roots, determinant, etc.) for the calculation of ρ containing as operands only the sequence of natural**

numbers, like
Finden Sie einen infiniten Ausdruck (wie z.B. Reihe, Produkt, Kettenbruch, geschachtelte Produkte oder Wurzeln, Determinante etc.) zur Berechnung von ρ, der als Operanden nur die Folge der natürlichen Zahlen enthält, wie

$$e^2 = \sum_{i=0}^{\infty} \frac{2^i}{i!} \; ; \; \frac{\pi^2}{6} = \sum_{i=1}^{\infty} \frac{1}{i^2}$$

Ch 4, equs (8.1) and (8.2) .

Ch 7
Selbstkorrektur?
Wir betrachten die “innere Struktur“ der Dezimaldarstellung transzendenter mathematischer Konstanten
e = 2.718281828…
pi = 3.141592654…
etc.
Ist eine Verfälschung
* einer Digitalstelle (Einfachfehler)
* von k aufeinanderfolgenden Digitalstellen (Bündelfehler)
in beliebiger Position

mittels eines Algorithmus, der allein die der Fehlerstelle\n benachbarten Dezimalstellen als Input einbezieht,
** erkennbar
** korrigierbar?
Mit anderen Worten: enthalten diese Konstanten in Dezimaldarstellung einen verteilten Selbstkorrekturmechanismus (vgl. fehlerkorrigierende Kodes in Informatik und Nachrichtentechnik; Selbstreparatur im genetischen Kode)?

Konstruieren Sie eine infinite Dezimalzahl, bei der für alle Stellen eine solche Einfehlerkorrektur mit Sicherheit realisiert werden kann.