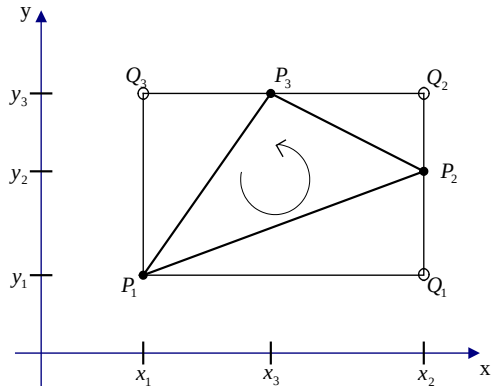
Small challenges to go, part 1
Ch 1
Visual resonance

A suitable 2D representation (matrix product, determinant, permanent, …) of a 1D (string) notated formula reveals often an unexpectedly simple "inner structure". Simple means here:

** visually convincing, mental resonance, beauty
** extensible
** transferable
** synergetic, broadband merging is possible.

Example 1

We start with a simple geometric problem.



Given is a planar triangle $\triangle P_1P_2P_3$ by the coordinates of its vertices $P_i = (x_i, y_i)$; $i = 1(1)3$, the simplest interaction between the points P_1, P_2 and P_3.

Find a formula expression for the area of $\triangle P_1P_2P_3$ in function of the coordinates x_i and y_i taking a simplest and shortest way.

Solution by embedding of $\triangle P_1P_2P_3$ in the minimal rectangle $Q_1Q_2Q_3Q_4$

$$\text{area}\Delta P_1P_2P_3 = \text{area}\square P_1Q_1Q_2Q_3 - \text{area}\Delta P_1Q_1P_2 - \text{area}\Delta P_2Q_2P_3 - \text{area}\Delta P_3Q_3P_1$$

$$= (x_2 - x_1)(y_3 - y_1) - \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$$

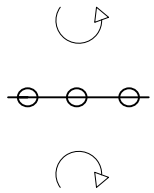
$$-\frac{1}{2}(x_2 - x_3)(y_3 - y_2) - \frac{1}{2}(x_3 - x_1)(y_3 - y_1)$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3) \qquad (1)$$

1D string notation (note the high inner symmetry in this expression!).

The classical 2D determinant notation of equ. (1) is a much more convincing reflection of the given geometric object:

$$area_{\triangle P_1P_2P_3} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff}$$

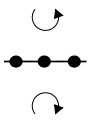


$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_1 & x_2 \\ y_1 & y_2 & y_3 & y_1 & y_2 \end{vmatrix} \quad \begin{matrix} \} \\ \} \end{matrix} \quad \begin{matrix} \text{redundancy} \\ \text{information} \end{matrix} \qquad (2)$$

$P_1 \quad P_2 \quad P_3$

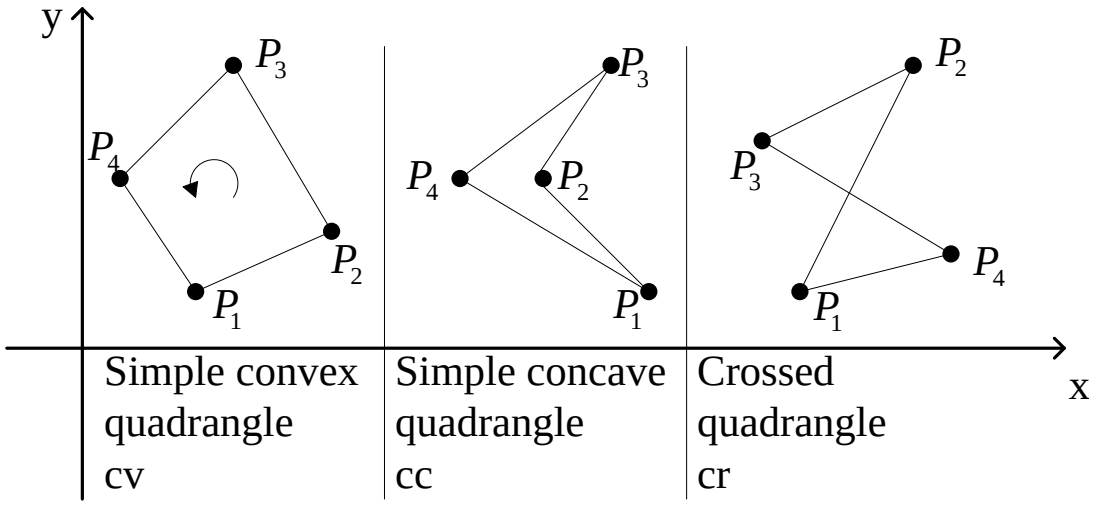
A very important engineering principle is: cancel unnecessary redundancy in your design . We go this way and cut the redundancy in equ.(2), getting the "shortened 2 by 3 determinant " notation

$$area_{\triangle P_1P_2P_3} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff} \qquad (3)$$



Formula design according to “creative visual resonance”.

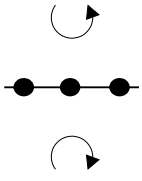
Let’s go to a first extension step from triangle with the result equs (1,2,3) to the quadrangle in 2D.



We start with the simple convex quadrangle cv

$$area_{\square}P_1P_2P_3P_4 = area_{\triangle}P_1P_2P_3 + area_{\triangle}P_1P_3P_4$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix} \quad \begin{matrix} >0 \\ =0 \\ <0 \end{matrix} \quad \text{iff} \quad \text{ \& ?} \quad (4)$$

A convincing extension!

Immediately extensible to n-gon!

By the way, in classical determinant notation:

$$= \frac{1}{2} \left| \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} \right| \begin{matrix} \} \text{redundancy} \\ \\ \} \text{information} \end{matrix} \quad (5)$$

Find similar classical determinant representations for the area of a simple convex 2D 5-gon, 6-gon, …
More can you find in Unit 19.

Now your exam task.

$$C = \left(x_c = \frac{1}{3}(x_1 + x_2 + x_3),\ y_c = \frac{1}{3}(y_1 + y_2 + y_3) \right) \quad (6)$$

is the centroid of $\Delta P_1P_2P_3$.

Find a shortened determinant representation for the tangent of the angle $\measuredangle P_3\underline{C}P_2$.

Compare your result with the shortened determinant representation of $\tan\measuredangle P_3\underline{P_1}P_2$.

Discussion!

Ch 2
Visual resonance

A suitable 2D representation (matrix product, determinant, permanent, …) of a 1D (string) notated formula reveals often an unexpectedly simple “inner structure”. Simple means here:

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** extensible
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** synergetic, broadband merging is possible.

Example 2

Differential calculus, quotient rule

Given

$$f(x) = \frac{u(x)}{v(x)} = \frac{1}{v(x)} * u(x) \qquad (1)$$

$$f'(x) = \frac{-1}{(v(x))^2} * [u(x)v'(x) - u'(x)v(x)] \qquad (2)$$

Find higher derivatives of $f(x)$.

Solution

Using product and quotient rules we obtain

$$f''(x) = \frac{1}{(v(x))^3} * [-u(x)v(x)v''(x) + 2u(x)v'(x)v'(x) - 2u'(x)v(x)v'(x) + u''(x)v(x)v(x)] \qquad (3)$$

$$f''' = \frac{-1}{v^4} * [uvvv''' - 6uvv'v'' + 6uv'v'v' + 3u'vvv'' - 6u'vv'v' + 3u''vvv' - u'''vvv] \qquad (4)$$

$$f^{////} = \frac{(-1)^4}{v^{4+1}} * [-uvvvv^{////} + 8uvvv^{/}v^{///} + 6uvvv^{//}v^{//}$$
$$-36uvv^{/}v^{/}v^{//} + 24uvv^{/}v^{/}v^{/}$$
$$-4u^{/}vvvv^{///} + 24u^{/}vvv^{/}v^{//}$$
$$-24u^{/}vv^{/}v^{/}v^{/} - 6u^{//}vvvv^{//}$$
$$+12u^{//}vvv^{/}v^{/} - 4u^{///}vvvv$$
$$+u^{////}vvvv] \qquad (5)$$

We can see

** the higher derivatives are unbalanced regarding u and v,

** the 1D formula representation of the higher derivatives is more and more unclear and not in “visual resonance”.

The situation is quiet different, however, with respect to the following 2D determinant representations. The structure of the solution is here very clear and in visual resonance.

$$f' = \frac{(-1)^1}{v^{1+1}} * \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \tag{6}$$

$$f'' = \frac{(-1)^2}{v^{2+1}} * \begin{vmatrix} u & 1v & 0 \\ u' & 1v' & 1v \\ u'' & 1v'' & 2v' \end{vmatrix} \tag{7}$$

$$f''' = \frac{(-1)^3}{v^{3+1}} * \begin{vmatrix} u & 1v & 0 & 0 \\ u' & 1v' & 1v & 0 \\ u'' & 1v'' & 2v' & 1v \\ u''' & 1v''' & 3v'' & 3v' \end{vmatrix} \tag{8}$$

$$f'''' = \frac{(-1)^4}{v^{4+1}} * \begin{vmatrix} u & 1v & 0 & 0 & 0 \\ u' & 1v' & 1v & 0 & 0 \\ u'' & 1v'' & 2v' & 1v & 0 \\ u''' & 1v''' & 3v'' & 3v' & 1v \\ u'''' & 1v'''' & 4v''' & 6v'' & 4v' \end{vmatrix} \tag{9}$$

Lo and behold, inside Pascal's triangle!
And, det(6)/det(7)/det(8)/… is the leading principal minor of det(7)/det(8)/det(9)/…!

Simplification of the determinant solution by normalization

Using the normalized derivative operator notation

$$F^{(i)} = \frac{\frac{d^i f(x)}{dx^i}}{i!} = \frac{f^{(i)}(x)}{i!} \; ; F = f(x) \qquad (10)$$

we obtain for the equs (6) - (9), respectively, the reduced forms without numeric coefficients, so for equ. (9)

$$F'''' = \frac{(-1)^4}{V^{4+1}} * \begin{vmatrix} U & V & 0 & 0 & 0 \\ U' & V' & V & 0 & 0 \\ U'' & V'' & V' & V & 0 \\ U''' & V''' & V'' & V' & V \\ U'''' & V'''' & V''' & V'' & V' \end{vmatrix} \qquad (11.1)$$

$$F'''' = \frac{(-1)^4}{V^{4+1}} * \begin{vmatrix} U & U' & U'' & U''' & U'''' \\ V & V' & V'' & V''' & V'''' \\ 0 & V & V' & V'' & V''' \\ 0 & 0 & V & V' & V'' \\ 0 & 0 & 0 & V & V' \end{vmatrix} \qquad (11.2)$$

Equ. (11.2) opens the door for a very innovative determinant representation of the McLaurin series of

$$f(x) = \frac{u(x)}{v(x)} \text{ with } v(0) = 1 \qquad (12.1)$$

$$f(x) = \frac{u(x)}{v(x)} = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = \sum_{i=0}^{\infty} F^{(i)}(0) x^i = \qquad (12.2)$$

$$- \begin{vmatrix} 0 & U(0) & U'(0) & U''(0) & U'''(0) & U^{(4)}(0) & \rightarrow \\ x^0 & 1 & V'(0) & V''(0) & V'''(0) & V^{(4)}(0) & \\ x^1 & 0 & 1 & V'(0) & V''(0) & V'''(0) & \\ x^2 & 0 & 0 & 1 & V'(0) & V''(0) & \\ x^3 & 0 & 0 & 0 & 1 & V'(0) & \\ x^4 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow \searrow & & & & & & \end{vmatrix}$$

(12.3)

In the following we take a look at three special cases

2.1

u(x)=sinx; v(x)=(sinx)'=cosx (14.1)

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$= \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\searrow & & & & & & \end{vmatrix}$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$

$$= \sum_{i=1}^{\infty} \frac{2^{2i}(2^{2i}-1)}{(2i)!}(-1)^{i-1}B_{2i}x^{2i-1} \qquad (14.2)$$

$$-\frac{\pi}{2} < x < +\frac{\pi}{2}$$

Equ. (14.2) generates immediately a 2D determinant representation of the Bernoulli numbers [1,2]

$$B_{2i} = (-1)^{i-1} \frac{(2i)!}{2^{2i}(2^{2i}-1)} \begin{vmatrix} \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \downarrow & & & & & \searrow \end{vmatrix}.$$

determinant of i-th order

(15)

More you can find in Unit 6, Unit 20 and Ch 3.

2.2

$$u(x) = x \; ; \; v(x) = \cos x \qquad (16.1)$$

$$f(x) = \frac{x}{\cos x} = \begin{vmatrix} x & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -x^3 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^5 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^7 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^9 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{11} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= x + \frac{1}{2}x^3 + \frac{5}{24}x^5 + \frac{61}{720}x^7 + \frac{1385}{40320}x^9 + \ldots .$$

$$= \sum_{i=0}^{\infty} \frac{1}{(2i)!}(-1)^i E_{2i} x^{2i+1} \; ; \; -\frac{\pi}{2} < x < +\frac{\pi}{2}$$

(16.2)

Equ. (16.2) generates immediately a 2D determinant representation of the Euler numbers [3,4]

$$E_{2i} = (-1)^i (2i)! \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \downarrow & & & & & \searrow \end{vmatrix} . \quad (17)$$

determinant of i-th order

An interesting lateral

The Gudermannian function [5,6]

$$\mathrm{gd}(x) = \arctan(\sinh x) = \tan^{-1}(\sinh x) \qquad (18.1)$$

$$\mathrm{gd}^{-1}(x) = \ln\left(\tan x + \frac{1}{\cos x}\right) \qquad (18.2)$$

$$\left.\begin{matrix}\mathrm{gd}(x)\\ \mathrm{gd}^{-1}(x)\end{matrix}\right\} = \begin{vmatrix} \frac{x^1}{1} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ \pm\frac{x^3}{3} & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ \frac{x^5}{5} & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ \pm\frac{x^7}{7} & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ \frac{x^9}{9} & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \pm\frac{x^{11}}{11} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\searrow & & & & & & \end{vmatrix}$$

$$= x \mp \frac{1}{6}x^3 + \frac{1}{24}x^5 \mp \frac{61}{5040}x^7 + \frac{1385}{362880}x^9 \mp \ldots$$

$$= \sum_{i=0}^{\infty} \frac{1}{(2i)!}(\pm 1)^i E_{2i} \frac{x^{2i+1}}{2i+1} \quad \text{for} \begin{cases} 0 \le \mathrm{abs}(x) < \infty \\ 0 \le \mathrm{abs}(x) < \dfrac{\pi}{2} \end{cases} \quad (18.3)$$

How about

$$
\begin{vmatrix}
\frac{x^1}{1!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\
\pm\frac{x^3}{3!} & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\
\frac{x^5}{5!} & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\
\pm\frac{x^7}{7!} & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\
\frac{x^9}{9!} & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\
\pm\frac{x^{11}}{11!} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\
\downarrow\ \searrow & & & & & &
\end{vmatrix} = ? \qquad (19)
$$

The comparison of the equs (15) and (17) shows the close structural connection between Bernoulli and Euler numbers.

Your task is to find a Pascal's triangle representation for B_{2i} and corresponding for E_{2i}.

Hint for solution:

$$\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = (2i+1=7)!*B_{2i=6} \quad (20)$$

$(2i=6)$ by $(2i=6)$ determinant

Find the way from equ. (15) to equ. (18).

References

[1] http://mathworld.wolfram.com/BernoulliNumber.html

[2] http://en.wikipedia.org/wiki/Bernoulli_number

[3] http://mathworld.wolfram.com/EulerNumber.html

[4] http://en.wikipedia.org/wiki/Euler_number

[5] http://mathworld.wolfram.com/Gudermannian.html

[6] http://mathworld.wolfram.com/InverseGudermannian.html

2.3

Find infinite determinant and power series representations for

$$\tan^2 x,\ \frac{1}{\cos^2 x},\ \frac{x}{\sin x},\ \frac{(x+1)^k-(x-1)^k}{(x+1)^k+(x-1)^k};k\in\mathbb{N} \quad (19)$$

.

Ch 3

Visual resonance

A suitable 2D representation (matrix product, determinant, permanent, …) of a 1D (string) notated formula reveals often an unexpectedly simple "inner structure". Simple means here:

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** transferable
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Example 3

An e^x-1 power tower

The simple 1D string structure of the power series representation of e^x-1

$$e^x - 1 = \sum_{i=1}^{\infty} \frac{1}{i!}\, x^i \qquad (1)$$

is reflected in the simple 2D determinant representation

$$e^{x}-1=\begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & \rightarrow \\ \frac{x^{1}}{1!} & 1 & -1 & 0 & 0 & 0 & 0 & \\ \frac{x^{2}}{2!} & 0 & 1 & -1 & 0 & 0 & 0 & \\ \frac{x^{3}}{3!} & 0 & 0 & 1 & -1 & 0 & 0 & \\ \frac{x^{4}}{4!} & 0 & 0 & 0 & 1 & -1 & 0 & \\ \frac{x^{5}}{5!} & 0 & 0 & 0 & 0 & 1 & -1 & \\ \frac{x^{6}}{6!} & 0 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & & \searrow \end{vmatrix} \quad (2)$$

Next step, how about that e power

$$e^{e^x-1}-1=\frac{1}{1!}x^1+\frac{2}{2!}x^2+\frac{5}{3!}x^3+\frac{15}{4!}x^4$$

$$+\frac{52}{5!}x^5+\frac{203}{6!}x^6+\frac{877}{7!}x^7+\ldots$$

$$=\sum_{i=1}^{\infty}\frac{BL_i}{i!}x^i\ ;$$

$$\left\langle BL_i\right\rangle_{i=1}^{\infty}=\left\langle 1,2,5,15,52,203,877,\ldots\right\rangle$$

Bell numbers [5,6] ? (3)

Surprisingly, we find a very clear determinant structure, inside Pascal's triangle (again the Pt!), in visual resonance!

$$e^{e^x-1}-1=\begin{vmatrix} 0 & -1 & -1 & -1 & -1 & -1 & -1 & \rightarrow \\ \frac{x^1}{1!} & 1 & -1 & -2 & -3 & -4 & -5 & \\ \frac{x^2}{2!} & 0 & 1 & -1 & -3 & -6 & -10 & \\ \frac{x^3}{3!} & 0 & 0 & 1 & -1 & -4 & -10 & \\ \frac{x^4}{4!} & 0 & 0 & 0 & 1 & -1 & -5 & \\ \frac{x^5}{5!} & 0 & 0 & 0 & 0 & 1 & -1 & \\ \frac{x^6}{6!} & 0 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & & \searrow \end{vmatrix} \quad (4)$$

The equs (3) and (4) provide a 2D determinant representation of the Bell numbers, so we have f.e.

$$
\mathrm{BL}_6 = +\begin{vmatrix}
-1 & -1 & -1 & -1 & -1 & -1 \\
1 & -1 & -2 & -3 & -4 & -5 \\
0 & 1 & -1 & -3 & -6 & -10 \\
0 & 0 & 1 & -1 & -4 & -10 \\
0 & 0 & 0 & 1 & -1 & -5 \\
0 & 0 & 0 & 0 & 1 & -1
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 2 & 3 & 4 & 5 \\
0 & -1 & 1 & 3 & 6 & 10 \\
0 & 0 & -1 & 1 & 4 & 10 \\
0 & 0 & 0 & -1 & 1 & 5 \\
0 & 0 & 0 & 0 & -1 & 1
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 \\
1 & 2 & 1 & -1 & 0 & 0 \\
1 & 3 & 3 & 1 & -1 & 0 \\
1 & 4 & 6 & 4 & 1 & -1 \\
1 & 5 & 10 & 10 & 5 & 1
\end{vmatrix} = 203 \qquad (5)
$$

$$\mathrm{BL}_7 = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & -1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & -1 & 0 \\ 1 & 5 & 10 & 10 & 3 & 1 & -1 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{vmatrix} = 877 \qquad (6)$$

Bell and Bernoulli numbers are connected via Pascal's triangle interface, see

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & 0 & 0 \\ 0 & 1 & 4 & 6 & 4 & 0 & 0 \\ 0 & 1 & 5 & 10 & 10 & 5 & 0 \\ 0 & 1 & 6 & 15 & 20 & 15 & 6 \\ 0 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} = 120 = 7! * \frac{1}{42}$$

$$= 7! * \text{Bernoulli}_6 \qquad (7)$$

Your task 1 is to extend the equs (2) and (4), that means to find corresponding infinite determinant representations for the power towers

$$e^{(e^{(e^x-1)}-1)}, \; e^{(e^{(e^{(e^x-1)}-1)}-1)}, \; \ldots \qquad (8)$$

Task 2

In example 2.1, equ. (14) we found

$$\left.\begin{aligned}\tan x &= \frac{\sin x}{\cos x}\\ \tanh x &= \frac{\sinh x}{\cosh x}\end{aligned}\right\} =$$

$$= \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ \mp x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ +x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ \mp x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ +x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \mp x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow\searrow & & & & & & \end{vmatrix}$$

$$= x \pm \frac{1}{3}x^3 + \frac{2}{15}x^5 \pm \frac{17}{315}x^7 + \frac{62}{2835}x^9 \pm \ldots$$

(9)

Find a determinant representation similar to equ. (4).

$$\left.\begin{aligned}\tan x &= \frac{\sin x}{\cos x}\\ \tanh x &= \frac{\sinh x}{\cosh x}\end{aligned}\right\} =$$

$$= \begin{vmatrix} 0 & ? & ? & ? & ? & ? & \rightarrow \\ \frac{\mp x^1}{1!} & 1 & ? & ? & ? & ? & \\ \frac{+x^3}{3!} & 0 & 1 & ? & ? & ? & \\ \frac{\mp x^5}{5!} & 0 & 0 & 1 & ? & ? & \\ \frac{+x^7}{7!} & 0 & 0 & 0 & 1 & ? & \\ \frac{\mp x^9}{9!} & 0 & 0 & 0 & 0 & 1 & \\ \downarrow\searrow & & & & & & \end{vmatrix}$$

$$= x \pm \frac{1}{3}x^3 + \frac{2}{15}x^5 \pm \frac{17}{315}x^7 + \frac{62}{2835}x^9 \pm \ldots$$

(10)

References

[1] http://mathworld.wolfram.com/BernoulliNumber.html
[2] http://en.wikipedia.org/wiki/Bernoulli_number
[3] http://mathworld.wolfram.com/EulerNumber.html
[4] http://en.wikipedia.org/wiki/Euler_number
[5] http://mathworld.wolfram.com/BellNumber.html
[6] http://en.wikipedia.org/wiki/Bell_number
[7] http://mathworld.wolfram.com/GeneratingFunction.html
[8] http://en.wikipedia.org/wiki/Generating_function

Lateral

[9]
http://www.spiegel.de/wissenschaft/mensch/mathematiker-faltings-kritisiert-mittelmass-an-deutschen-hochschulen-a-924142.html
„Man kann nicht sagen, das ist ein wichtiges Problem, und das will ich jetzt lösen. Ich muss auch mathematische Methoden haben, mit denen das gelingt. An den meisten großen Problemen haben sich schon viele kluge Leute versucht - und man sollte nicht unbedingt damit rechnen, dass man klüger ist als die. Die einzige Chance ist, ein mathematisches Werkzeug zur Hand zu nehmen, das die anderen noch nicht hatten. Wenn ich ein solches neues Werkzeug hätte, würde ich das Problem lösen.“
[10] http://en.wikipedia.org/wiki/Visual_analytics
[11] http://en.wikipedia.org/wiki/Data_mining
[12] http://de.wikipedia.org/wiki/Synergetik