In the debating club n° 43

$$\frac{e}{\pi}$$

43.1 A weak convergent matrix recurrence equation [1]

$$\begin{pmatrix} u(i+2) \\ v(i+2) \end{pmatrix} = \begin{pmatrix} u(i+1) & u(i) \\ v(i) & v(i+1) \end{pmatrix} \begin{pmatrix} 1 \\ i^{-1} \end{pmatrix};$$

$$i = 2,3,4,... \tag{1.1}$$

$$\begin{pmatrix} u(3) & u(2) \\ v(2) & v(3) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.2}$$

$$\lim_{i \to \infty} \frac{(v(i))^2}{u(i)} = \frac{2e}{\pi} \tag{1.3}$$

Record 1

i	$u(i)$	$v(i)$	$\frac{(v(i))^2}{u(i)}$
2	1	1	1
3	1	1	1
4	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

5	$\frac{11}{6}$	$\frac{3}{2}$	$\frac{27}{11}$
6	$\frac{53}{24}$	$\frac{15}{8}$	$\frac{675}{424}$
7	$\frac{103}{40}$	$\frac{15}{8}$	$\frac{1125}{824}$
8	$\frac{2119}{720}$	$\frac{35}{16}$	$\frac{55125}{33904}$
9	$\frac{16687}{5040}$	$\frac{35}{16}$	$\frac{385875}{266992}$

Record 2, decimal representation

i	$u(i)$ $v(i)$	$\frac{(v(i))^2}{u(i)}$
2		1.0000000000000000000
3		1.0000000000000000000
4		1.5000000000000000000
5		1.2272727272727272727
6		1.5919811320754716981
7		1.3652912621359223301
8		1.6259143463898065125
9		1.4452680230119254509
10		1.6462493742794733329
11		1.4965904421699103208
12		1.6599682226866535606

13	1.532278360071950405 3
14	1.669848986593803463 3
15	1.558525720824122962 6
16	1.677303669572937378 2
17	1.578638747833364891 0
18	1.683127640647855559 2
19	1.594541975350600039 1
20	1.687802995205210749 4
21	1.607431424004962618 5
22	1.691638911103404410 2
23	1.618089393229343348 9
24	1.694842772677463888 3
25	1.627049061770365332 8
26	1.697558866864447003 5
27	1.634686316239837855 2
28	1.699890678494755309 8
$2\frac{e}{\pi}$	1.730511958864530174 4

Converges very slowly. → use the

tripel accelerator $\left\langle f(t)\right\rangle_{t=0}^{\infty} \rightarrow \left\langle f1(t)\right\rangle_{t=0}^{\infty}$ with

$$f1(t) = \frac{\begin{vmatrix} f(t) & f(t+1) \\ f(t+1) & f(t+2) \end{vmatrix}}{f(t) - 2f(t+1) + f(t+2)} \qquad (2.1)$$

Test the performance of equ. (2.1) for one-time application.
What happens for repeated application of the tripel accelerator:

$$\langle f(t)\rangle_{t=0}^{\infty} \rightarrow \langle f1(t)\rangle_{t=0}^{\infty} \rightarrow \langle f2(t)\rangle_{t=0}^{\infty} \rightarrow \ldots$$

$$f2(t) = \frac{\begin{vmatrix} f1(t) & f1(t+1) \\ f1(t+1) & f1(t+2) \end{vmatrix}}{f1(t) - 2f1(t+1) + f1(t+2)} \qquad (2.2)$$

Transform the recurrence equation (1) + tripel accelerator equ. (2) into a fast converging r.e.!
Examine the following extension of equ. (1)

$$\begin{pmatrix} u(i+2) \\ v(i+2) \end{pmatrix} = \begin{pmatrix} u(i+1) & u(i) \\ v(i) & v(i+1) \end{pmatrix} \begin{pmatrix} a \\ bi^{-1} \end{pmatrix};$$

$$i = 2,3,4,\ldots \qquad (3.1)$$

$$\begin{pmatrix} u(3) & u(2) \\ v(2) & v(3) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad (3.2)$$

$$\lim_{i\to\infty} \frac{(v(i))^2}{u(i)} = ?? \qquad (3.3)$$

starting with a=1, b=2.
Find a similar recurrence equation for $\sqrt{2e/\pi}$.

43.2 Infinite product representation of $\frac{2e}{\pi}$ and $\sqrt{\frac{2e}{\pi}}$ [2]

$$\frac{2e}{\pi} = \left(2^{\binom{1}{1}}\right)^{g(1)} \left(2^{\binom{2}{1}}3^{-\binom{2}{2}}\right)^{g(2)} \left(2^{\binom{3}{1}}3^{-\binom{3}{2}}4^{\binom{3}{3}}\right)^{g(3)}$$

$$\left(2^{\binom{4}{1}}3^{-\binom{4}{2}}4^{\binom{4}{3}}5^{-\binom{4}{4}}\right)^{g(4)} \left(2^{\binom{5}{1}}3^{-\binom{5}{2}}4^{\binom{5}{3}}5^{-\binom{5}{4}}6^{\binom{5}{5}}\right)^{g(5)} = \ldots$$

$$= \prod_{n=1}^{\infty}\left(\prod_{k=1}^{n}(k+1)^{(-1)^{(k+1)}\binom{n}{k}}\right)^{g(n)}$$

$$g(n) = n^{-1} - 2^{-n} \qquad (4)$$

$$\frac{2e}{\pi} = 1.730511958864530174 4...$$

$$\infty \to 27 \; 1.716697625484386812 5...$$

Converges very slowly.

$$\sqrt{\frac{2e}{\pi}} = \left(2^{-2\binom{2}{1}}3^{3\binom{2}{2}}\right)^{\frac{1}{3}}\left(2^{-2\binom{3}{1}}3^{3\binom{3}{2}}4^{-4\binom{3}{3}}\right)^{\frac{1}{4}}$$

$$\left(2^{-2\binom{4}{1}}3^{3\binom{4}{2}}4^{-4\binom{4}{3}}5^{5\binom{4}{4}}\right)^{\frac{1}{5}}\left(2^{-2\binom{5}{1}}3^{3\binom{5}{2}}4^{-4\binom{5}{3}}5^{5\binom{5}{4}}6^{-6\binom{5}{5}}\right)^{\frac{1}{6}} = \ldots$$

$$= \prod_{n=2}^{\infty}\left(\prod_{k=1}^{n}(k+1)^{(-1)^k(k+1)\binom{n}{k}}\right)^{\frac{1}{n+1}} \qquad (5)$$

$$\sqrt{\frac{2e}{\pi}} = 1.3154892469589138281...$$

$\infty \rightarrow 24$ 1.3124589515930453674...

Converges very slowly.

Check the convergence of the equs (4) and (5).
Find fast converging recurrence equations basing on the equs (4) and (5).

References

[1] http://www.pi314.net/eng/miroir.php

[2] J. Guillera, J. Sondow. Double integrals and infinite products for some classical constants via analytic continuations of Lerch's transcendent. arXiv:math/0506319v3 [math.NT]

Appendix 1

Check the following candidates for better accelerators

$\langle f(t)\rangle_{t=0}^{\infty} \rightarrow \langle f1(t)\rangle_{t=0}^{\infty}$ with

$$f1(t) = \frac{\begin{vmatrix} f(t) & f(t+1) \\ f(t+1) & f(t+2) \end{vmatrix}}{f(t) - 2f(t+1) + f(t+2)} \qquad (2.1)$$

$\rightarrow$ extensions?

$$f1(t) = \frac{\begin{vmatrix} f(t) & f(t+1) & f(t+2) \\ f(t+1) & f(t+2) & f(t+3) \end{vmatrix}}{f(t) - 3f(t+1) + 3f(t+2) - f(t+3)} \qquad (2.3)$$

or

$$f1(t) = \frac{\begin{vmatrix} f(t) & f(t+1) & f(t+2) \\ f(t+1) & f(t+2) & f(t+3) \\ f(t+2) & f(t+3) & f(t+4) \end{vmatrix}}{\begin{vmatrix} f(t) & f(t+1) \\ f(t+1) & f(t+2) \end{vmatrix}} \qquad (2.4)$$

or ???