Im Debattierclub 20
Infinite formula patterns (power series, nested expressions, continued fractions, …) and their mirror images.
Finger exercises, just for fun or a rice field?
20.1

$$\sum_{i=0}^{\infty}\binom{2i}{i}\left(\frac{x}{4}\right)^{i} =$$

$$1+\frac{1}{2}x\left(1+\frac{3}{4}x\left(1+\frac{5}{6}x\left(1+\frac{7}{8}x\left(1+\frac{9}{10}x\left(\quad ...\right.\right.\right.\right.\right.$$

(1)

$$\sum_{i=0}^{\infty}\frac{1}{\binom{2i}{i}}(4x)^{i} =$$

$$1+\frac{2}{1}x\left(1+\frac{4}{3}x\left(1+\frac{6}{5}x\left(1+\frac{8}{7}x\left(1+\frac{10}{9}x\left(\quad ...\right.\right.\right.\right.\right.$$

(2)

Find the continued fraction representations for equ. (1) and equ. (2) (see unit 61).

20.2

$$e-1 = 1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cfrac{6}{6+\cfrac{7}{7+\ldots}}}}}} \qquad (3.1)$$

For comparision

$$\sqrt{3} = 1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\ldots}}}}} \qquad (3.2)$$

20.3

$$\frac{3-e}{e-2} = \cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\cfrac{5}{6+\cfrac{6}{7+\ldots}}}}}} \qquad (4)$$

$$1 = \cfrac{2}{1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{5+\cfrac{7}{6+\ldots}}}}}} \qquad (5)$$

20.4

$$\frac{2-\sqrt{e}}{\sqrt{e}-1} = \cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9+\cfrac{10}{11+\cfrac{12}{13+\ldots}}}}}}$$

(6)

$$1 = \cfrac{3}{2+\cfrac{5}{4+\cfrac{7}{6+\cfrac{9}{8+\cfrac{11}{10+\cfrac{13}{12+\ldots}}}}}}$$

(7)

20.5 Lateral 1: comparision e - π

$$\frac{1}{e-2} = 1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cfrac{5}{6 + \cfrac{6}{7 + \ldots}}}}}} \quad (8)$$

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \cfrac{5^2}{11 + \cfrac{6^2}{13 + \ldots}}}}}} \quad (9)$$

20.6 Lateral 2: scf of e-expressions

The simple continued fractions (scf)

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} = \ldots + \cfrac{1}{a_n}$$

$$[a_0; a_1, a_2, a_3, \ldots, a_n] \qquad (10)$$

of the e-expressions equs (3), (4), (6), and (8) have all the same structure, a systematically structured sequence of information-bearing elements and mirror image arranged "bodyguards" 1.

$$e-1=[1;1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,\ldots]$$

$$\frac{3-e}{e-2}=[0;2,1,1,4,1,1,6,1,1,8,1,1,10,1,\ldots]$$

$$\frac{2-\sqrt{e}}{\sqrt{e}-1}=[0;1,1,5,1,1,9,1,1,13,1,1,17,1,\ldots]$$

$$\frac{1}{e-2}=[1;2,1,1,4,1,1,6,1,1,8,1,1,10,1,\ldots]$$

$$(11.1-4)$$

More:

$$\frac{\sqrt[8]{e}}{3-2*\sqrt[8]{e}}=$$

[1; 1, 1, 5, 7, 1, 5, 13, 5, 1, 17, 1, 5, 23, 1, 5,
29, 5, 1, 33, 1, 5, 39, 1, 5, 45, 5, 1, 49, 1, 5,
55, 1, 5, 61, 5, 1, 65, 1, 5, 71, 1, 5, 77, 5, 1,
81, 1, 5, 87, 1, 5, 93, 5, 1, 97, 1, 5, 103, 1, 5,
109, 5, 1, 113, 1, 5, 119, 1, 5, 125, 5, 1,
129, 1, 5, 135, 1, 5, 141, 5, 1, 145, 1, 5,
151, 1, 5, 157, 5, 1, 161, 1, 5, 167, 1, 5,
173, 5, 1, 177, 1, 5, 183, 1, 5, 189, 5, 1,
193, 1, 5, 199, 1, 5, 205, 5, 1, 209, …]

$$\frac{\sqrt[8]{e}}{3+2*\sqrt[8]{e}} = [0;\ 4,\ 1,$$

1, 1, 5, 7, 1, 5, 13, 5, 1, 17, 1, 5, 23, 1, 5,
29, 5, 1, 33, 1, 5, 39, 1, 5, 45, 5, 1, 49, 1, 5,
55, 1, 5, 61, 5, 1, 65, 1, 5, 71, 1, 5, 77, 5, 1,
81, 1, 5, 87, 1, 5, 93, 5, 1, 97, 1, 5, 103, 1, 5,
109, 5, 1, 113, 1, 5, 119, 1, 5, 125, 5, 1,
129, 1, 5, 135, 1, 5, 141, 5, 1, 145, 1, 5,
151, 1, 5, 157, 5, 1, 161, 1, 5, 167, 1, 5,
173, 5, 1, 177, 1, 5, 183, 1, 5, 189, 5, 1,
193, 1, 5, 199, 1, 5, 205, 5, 1, 209, …]

(12.1,2)

Bodyguards 1 and 5, in sequence 15,15,51.

See also Deb 21.

20.7 Lateral 3: An open bodyguard problem

e = 2.7182818284…= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1,…]

How about

[1; 2, 1, 4, 1, 6, 1, 8, 1,…]

[1; 1, 1, 2, 1, 1, 1, 4, 1, 1, 1, 6, 1, 1, 1, 8, 1,…]

[1; 1, 1, 1, 2, 1, 1, 1, 1, 4, 1, 1, 1, 1, 6, 1, 1, 1, 1, 8, 1,…] **?**

(14.1-4)

20.8 Lateral 4: Other scf with bodyguards

$e = [2;1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,\ldots]$

$e^2 =$

$[7;2,1,1,3,18,5,1,1,6,30,8,1,1,9,42,11,1,1,\ldots]$

$\tan(1) = [1;1,1,3,1,5,1,7,1,9,1,11,1,13,1,\ldots]$

$$\frac{1+\sqrt{5}}{2} = [1;1,1,1,1,1,1,1,1,1,1,1,1,1,\ldots]$$

$$\frac{1+\sqrt{21}}{2*5} = [1;1,1,3,1,3,1,3,1,3,1,3,1,3,1,\ldots]$$

$$\frac{121+\sqrt{45}}{2*41} = [1;1,1,3,1,5,1,5,1,5,1,5,1,5,1,\ldots]$$

$$\frac{3539+\sqrt{77}}{2*1139} = [1;1,1,3,1,5,1,7,1,7,1,7,1,7,1,\ldots]$$

$[1;1,1,3,1,5,1,7,1,\ldots,k,1,k,1,k,1,k,1,\ldots] =$

$$\frac{A(k)+\sqrt{k(k+4)}}{2*B(k)}$$

Find an algorithm for calculation of A(k) and B(k) (recurrence equ., determinant, 2by2 matrix chain product, …).

$$\cfrac{3}{1+\cfrac{7}{5+\cfrac{11}{9+\cfrac{15}{13+\cfrac{19}{17+\cfrac{23}{21+\cfrac{27}{25+\cfrac{31}{29+\cfrac{35}{33+\ldots}}}}}}}}}}$$

$= 1.396045039671837489058645721659497782916225005279628092257529484556130032916821877300113762092137716\ldots$

$= [1; 2, 1, 1, 9, 1, 1, 17, 1, 1, 25, 1, 1, 33, 1, 1, 41, 1, 1, 49, 1, 1, 57, 1, 1, 65, 1, 1, 73, 1, 1, 81, 1, 1, 89, 1, 1, 97, 1, 1, 105, 1, 1, 113, 1, 1, 121, 1, 1, 129, 1, 1, 137, 1, 1, 145, 1, 1, 153, 1, 1, 161, 1, 1, 16, 1, 1, 177, 1, 1, 185, 1, 1, 193, 1, 1, 201, 1, 1, 209, 1, 1, 217, 1, 1, 225, 1, 1, 233, 1, 1, 241, 1, 1, \ldots]$

$= f(e)$???

$$\cfrac{3}{1+\cfrac{5}{3+\cfrac{7}{5+\cfrac{9}{7+\cfrac{11}{9+\cfrac{13}{11+\cfrac{15}{13+\cfrac{17}{15+\cfrac{19}{17+\ldots}}}}}}}}}}$$

$= 1.359801999955458725565369924206688423801659158365668336889567081809920093200091343 36...$

$= [1; 2, 1, 3, 1, 1, 7, 1, 1, 11, 1, 1, 15, 1, 1, 19, 1, 1, 23, 1, 1, 27, 1, 1, 31, 1, 1, 35, 1, 1, 39, 1, 1, 43, 1, 1, 47, 1, 1, 51, 1, 1, 55, 1, 1, 59, 1, 1, …]$

$= g(e)$???

(15.1-10)

Find more.

20.9 Lateral 5: Infinite zip nested expressions (izne)

$$\sqrt{2} = 1+\frac{2}{5}(1+\frac{1}{29}(1+\frac{5}{169}(1+\frac{29}{985}(1+\frac{169}{5741}(1+\ldots$$

$$\sqrt{3} = 1+\frac{2}{3}(1+\frac{1}{11}(1+\frac{3}{41}(1+\frac{11}{153}(1+\frac{41}{571}(1+\ldots$$

$$\sqrt{5} = 2+\frac{4}{17}(1+\frac{1}{305}(1+\frac{17}{5473}(1+\frac{305}{98209}(1+\ldots$$

$$\sqrt{17} = 4+\frac{8}{65}(1+\frac{1}{4737}(1+\frac{65}{312577}(1+\frac{4737}{?}(1+\ldots$$

(16.1-4)

More!

Find

**** the defining equations for the iznes equ. (16),**

**** an algorithm** $\sqrt{\text{prime}} \rightarrow \text{izne}$.

Check and discuss the convergence of the iznes equ. (16).

20.10 Lateral 6: Transformation infinite continued fraction (icf) into infinite zip nested expression (izne)

$$z = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5 + \ldots}}}}} \in \mathbb{R}_+$$

$$\Rightarrow a_0 + b_1 \frac{1}{D_2}(a_2 + b_2 b_3 \frac{D_0}{D_4}(a_4 + b_4 b_5 \frac{D_2}{D_6}($$

$$a_6 + b_6 b_7 \frac{D_4}{D_8}(a_8 + b_8 b_9 \frac{D_6}{D_{10}}(a_{10} + \ldots$$

(17)

with

$$D_0 = 1$$

$$D_i = \begin{vmatrix} a_1 & b_2 & 0 & 0 & 0 & 0 & 0 \\ -1 & a_2 & b_2 & 0 & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_4 & b_2 & 0 & 0 \\ 0 & 0 & 0 & -1 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \ddots & a_{i-1} & b_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & a_i \end{vmatrix}$$

$$i=1,2,3,4,...$$

Recurrence formula

$$D_{2i} = \frac{1}{a_{2i-2}} \begin{vmatrix} D_{2i-2} & b_{2i-2} & 0 & 0 \\ 0 & a_{2i-2} & b_{2i-1} & 0 \\ D_{2i-4} & -1 & a_{2i-1} & b_{2i} \\ 0 & 0 & -1 & a_{2i} \end{vmatrix}$$

$$i=2,3,4,...$$

$$D_0 = 1\ ,\ D_2 = \begin{vmatrix} a_1 & b_2 \\ -1 & a_2 \end{vmatrix}$$

$$(18.1-3)$$

Derive the equs (17) and (18).

Check the example 1

$a_i = b_i = k \in \mathbb{N}$; i=1,2,3,4,… $a_0 = 0$ (19.1)

f.e.

k	$z = \frac{1}{2}(\sqrt{k(k+4)} - k) = [0;\overline{1,\ k,}]$
1	$\frac{1}{2}(\sqrt{5} - 1)$
2	$\sqrt{3} - 1$
3	$\frac{1}{2}(\sqrt{21} - 3)$
4	$\sqrt{8} - 2$
5	$\frac{1}{2}(\sqrt{45} - 5)$
⋮	

(19.2)

Check the example 2

$a_i = k \in \mathbb{N}$, $b_i = m \in \mathbb{N}$; i=1,2,3,4,… (20.1)

$a_0 = 0$

$$\rightarrow z = \frac{1}{2}(\sqrt{k^2 + 4m} - k)$$

= positive root of the quadratic equ.

$$z^2 + kz - m = 0 \qquad (20.2)$$

To be continued.