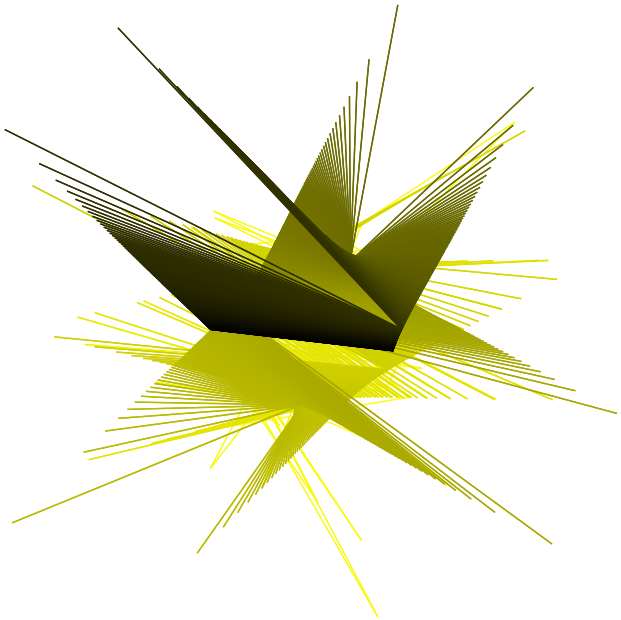
Unit 61

A determinant-based Maclaurin series and infinite continued fraction representation of real functions f(x), esp. $f(x) = \frac{u(x)}{v(x)}$.

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1. The problem

f(x) is a real analytic function represented by an equation.

$$F^{(i)} = \frac{f^{(i)}(0)}{i!} \; ; \; i=0,1,2,3,... \; ; \; F^{(0)} = F = f(0) \qquad (1)$$

denotes the normalized i-th derivative of f(x) evaluated at x=0.

The present talk shows the possibility of a reversible, consistent determinant-based Maclaurin series and infinite continued fraction representation of f(x), esp. $f(x) = \frac{u(x)}{v(x)}$.

2. A determinant-based Maclaurin series representation of f(x)

The Maclaurin series of f(x)

$$f(x) = \sum_{i=0}^{\infty} F^{(i)} x^i$$

$$= F^{(0)} + F^{(1)}x + F^{(2)}x^2 + F^{(3)}x^3 + F^{(4)}x^4 + \ldots \qquad (2)$$

can be represented by the infinite determinant

$$f(x) = -\begin{vmatrix} 0 & F & F^{(1)} & F^{(2)} & F^{(3)} & F^{(4)} & \rightarrow \\ 1 & 1 & 0 & 0 & 0 & 0 & \\ x & 0 & 1 & 0 & 0 & 0 & \\ x^2 & 0 & 0 & 1 & 0 & 0 & \\ x^3 & 0 & 0 & 0 & 1 & 0 & \\ x^4 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} . \qquad (3)$$

3. Extension of the determinant representation equ. (3) for f(x) = u(x)/v(x)

Equ. (3) is the suitable launch pad (see [1])
for the following extension:
The Maclaurin series of

$$f(x) = \frac{u(x)}{v(x)} \; ; \; v(0) = 1 \; ;$$

u(x) and v(x) are real analytic functions (4)
can be represented by the infinite determinant

$$f(x) = \frac{u(x)}{v(x)}$$

$$= -\begin{vmatrix} 0 & U & U^{(1)} & U^{(2)} & U^{(3)} & U^{(4)} & \rightarrow \\ 1 & 1 & V^{(1)} & V^{(2)} & V^{(3)} & V^{(4)} & \\ x & 0 & 1 & V^{(1)} & V^{(2)} & V^{(3)} & \\ x^2 & 0 & 0 & 1 & V^{(1)} & V^{(2)} & \\ x^3 & 0 & 0 & 0 & 1 & V^{(1)} & \\ x^4 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} . \quad (5)$$

4. Special cases of equs (4) and (5)

4.1

In the special case

$$u(x) = 1 \tag{6.1}$$

we obtain from the Maclaurin series of

$$v(x) = \sum_{i=0}^{\infty} V^{(i)} x^i \; ; \; v(0) = V = 1 \tag{6.2}$$

via equ. (5) the infinite power series of $\dfrac{1}{v(x)}$

in determinant representation

$$\frac{1}{v(x)} =$$

$$-\begin{vmatrix} 1 & V^{(1)} & V^{(2)} & V^{(3)} & V^{(4)} & V^{(5)} & \rightarrow \\ x & 1 & V^{(1)} & V^{(2)} & V^{(3)} & V^{(4)} & \\ x^2 & 0 & 1 & V^{(1)} & V^{(2)} & V^{(3)} & \\ x^3 & 0 & 0 & 1 & V^{(1)} & V^{(2)} & \\ x^4 & 0 & 0 & 0 & 1 & V^{(1)} & \\ x^5 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} . \tag{6.3}$$

4.2

For the special case

$$u(0) = u^{(2)}(0) = u^{(4)}(0) = u^{(6)}(0) = \ldots = 0 \ ;$$

$$v^{(1)}(0) = v^{(3)}(0) = u^{(5)}(0) = u^{(7)}(0) = \ldots = 0 \qquad (7.1)$$

equ. (5) can be reduced to

$$f(x) = \frac{u(x)}{v(x)} =$$

$$-\begin{vmatrix} 0 & U^{(1)} & U^{(3)} & U^{(5)} & U^{(7)} & U^{(9)} & \rightarrow \\ x & 1 & V^{(2)} & V^{(4)} & V^{(6)} & V^{(8)} & \\ x^3 & 0 & 1 & V^{(2)} & V^{(4)} & V^{(6)} & \\ x^5 & 0 & 0 & 1 & V^{(2)} & V^{(4)} & \\ x^7 & 0 & 0 & 0 & 1 & V^{(2)} & \\ x^9 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} . \qquad (7.2)$$

Example 1

Given $f(x) = \dfrac{u(x)}{v(x)}$ by an equation.

Find an infinite determinant representation and the corresponding Maclaurin series of f(x).

$u(x) = \sin x$; $v(x) = \cos x$; $f(x) = \tan x$ (8)

With equ. (8) we obtain a 2D infinite determinant representation for tanx

$$\tan x = \frac{\sin x}{\cos x} = \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix} \quad (9.1)$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$

$$-\frac{\pi}{2} < x < +\frac{\pi}{2} \quad . \quad (9.2)$$

What does the series in equ. (9) represent for

$\text{abs}(x) > \frac{\pi}{2}$?

Comparing the equs (9) with the canonical Maclaurin series of tanx

$$\tanx = \sum_{i=1}^{\infty} \frac{2^{2i}(2^{2i}-1)}{(2i)!}(-1)^{i-1} B_{2i} x^{2i-1} \; ; \; -\frac{\pi}{2} < x < +\frac{\pi}{2} \tag{10}$$

we get a 2D determinant representation of the Bernoulli numbers [2,3]

$$B_{2i} = (-1)^{i-1} \frac{(2i)!}{2^{2i}(2^{2i}-1)} \begin{vmatrix} \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \downarrow & & & & & \searrow \end{vmatrix}.$$

determinant of i-th order

(11)

Similar 2D determinant representations of B_{2i} are

$$B_{2i} = (-1)^{i-1} \frac{(2i)!}{(2^{2i}-2)} \begin{vmatrix} \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \frac{1}{11!} & \rightarrow \\ \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \\ 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \\ 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \\ 0 & 0 & 0 & \frac{1}{1!} & \frac{1}{3!} & \\ \downarrow & & & & & \searrow \end{vmatrix} \qquad (12)$$

determinant of i-th order

$$B_{2i} = \frac{(-1)^{i-1}}{2(2^{2i}-1)} *$$

$$\begin{vmatrix}
1 & \binom{1}{1} & 0 & 0 & 0 & 0 & 0 & \rightarrow \\
0 & \binom{2}{0} & \binom{2}{2} & 0 & 0 & 0 & 0 & \\
0 & 0 & \binom{3}{1} & \binom{3}{3} & 0 & 0 & 0 & \\
0 & 0 & \binom{4}{0} & \binom{4}{2} & \binom{4}{4} & 0 & 0 & \\
0 & 0 & 0 & \binom{5}{1} & \binom{5}{3} & \binom{5}{5} & 0 & \\
0 & 0 & 0 & \binom{6}{0} & \binom{6}{2} & \binom{6}{4} & \binom{6}{6} & \\
0 & 0 & 0 & 0 & \binom{7}{1} & \binom{7}{3} & \binom{7}{5} & \\
\downarrow & & & & & & & \searrow
\end{vmatrix} \qquad (13)$$

determinant of i-th order

Example 2

Given $f(x) = \dfrac{u(x)}{v(x)}$ by an equation.

Find an infinite determinant representation and the corresponding Maclaurin series of f(x).

$$u(x) = x \; ; \; v(x) = \cos x \; ; \; f(x) = \frac{x}{\cos x} \tag{14}$$

With equ. (14) we obtain a 2D infinite determinant representation for $\dfrac{x}{\cos x}$

$$\frac{x}{\cos x} = \begin{vmatrix} x & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ -x^3 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ x^5 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^7 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ x^9 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^{11} & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix} \tag{15.1}$$

$$= x + \frac{1}{2}x^3 + \frac{5}{24}x^5 + \frac{61}{720}x^7 + \frac{1385}{40320}x^9 + \ldots .$$

$$-\frac{\pi}{2} < x < +\frac{\pi}{2} \qquad (15.2)$$

Comparing equs (15) with the Maclaurin series of $\dfrac{x}{\cos x}$

$$\frac{x}{\cos x} = \sum_{i=0}^{\infty} \frac{1}{(2i)!}(-1)^i E_{2i} x^{2i+1} \; ; \; -\frac{\pi}{2} < x < +\frac{\pi}{2} \qquad (16)$$

we obtain a 2D determinant representation of the Euler numbers [4,5]

$$E_{2i} = (-1)^i (2i)! \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \frac{1}{10!} & \rightarrow \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ \downarrow & & & & & \searrow \end{vmatrix}. \qquad (17)$$

determinant of i-th order

The comparison of the equs (12) and (17) shows the close structural connection between Bernoulli and Euler numbers.
Find a Pascal's triangle representation for E_{2i} corresponding to B_{2i} equ. (14).

Another interesting comparison!
Pascal's triangle and the sum of powers of integers

$$\frac{\sum_{i=1}^{n} i^{6}}{\sum_{i=1}^{n} i^{0}} = \frac{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^{1} & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^{2} & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^{3} & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^{4} & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^{5} & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^{6} & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^{1} & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^{2} & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^{3} & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^{4} & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^{5} & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^{6} & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}} \qquad (18.1)$$

(6+1) by (6+1) determinants

$$\Delta(k=7,n)=\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 3 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 4 & 4 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 10 & 5 & 0 & 0 \\ p^4 & 0 & 0 & 6 & 20 & 6 & 0 \\ p^5 & 0 & 0 & 1 & 21 & 35 & 7 \\ p^6 & 0 & 0 & 0 & 8 & 56 & 56 \end{vmatrix}$$

7 by 7 determinant (18.2)

$$\frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^3} = \frac{\Delta(k,n)}{\Delta(k,1)}$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i^1 \right)^2 = \frac{p^2}{4} \; ; \; p=n(n+1) \qquad (18.3)$$

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{6}}{1}=\frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ n^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ n^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ n^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ n^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ n^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

(6+1) by (6+1) determinants

(18.4)

$$D(k=7,n)=\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ p^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ p^2 & 0 & 3 & 1 & 0 & 0 & 0 \\ p^3 & 0 & 1 & 6 & 1 & 0 & 0 \\ p^4 & 0 & 0 & 5 & 10 & 1 & 0 \\ p^5 & 0 & 0 & 1 & 15 & 15 & 1 \\ p^6 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant (18.5)

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i}i^{2}}=\frac{D(k,n)}{D(k,1)}$$

$$\sum_{i=1}^{n}(-1)^{n-i}i^{2}=\sum_{i=1}^{n}i^{1}=\frac{1}{2}p\ ;\ p=n(n+1) \qquad (18.6)$$

5. A determinant-based infinite continued fraction representation of f(x) for the special case $f^{(2k)}(0) = 0$; k=0,1,2,3,… $f^{(1)}(0) = 1$ corresponding with section 4.2

5.1 Transformation of the Maclaurin series

of this f(x) into an infinite continued fraction

The Maclaurin series of

f(x) with $f^{(2k)}(0) = 0 \rightarrow F^{(2k)} = 0\ ;\ k = 0,1,2,3,\ldots$

and $f^{(1)} = 1 \rightarrow F^{(1)} = 1$

$$= F^{(1)}x + F^{(3)}x^3 + F^{(5)}x^5 + F^{(7)}x^7 + F^{(9)}x^9 + \ldots$$

$$= -\begin{vmatrix} 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} & \rightarrow \\ x & 1 & 0 & 0 & 0 & 0 & \\ x^3 & 0 & 1 & 0 & 0 & 0 & \\ x^5 & 0 & 0 & 1 & 0 & 0 & \\ x^7 & 0 & 0 & 0 & 1 & 0 & \\ x^9 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix} \qquad (19)$$

can be reversibly transformed into an infinite simple continued fraction

$$f(x) = [0; \frac{\alpha_1}{x}, \frac{\alpha_2}{x}, \frac{\alpha_3}{x}, \frac{\alpha_4}{x}, \frac{\alpha_5}{x}, ...]$$

$$= \cfrac{1}{\cfrac{\alpha_1}{x} + \cfrac{1}{\cfrac{\alpha_2}{x} + \cfrac{1}{\cfrac{\alpha_3}{x} + \cfrac{1}{\cfrac{\alpha_4}{x} + \cfrac{1}{\cfrac{\alpha_5}{x} + ...}}}}} \qquad (20.1)$$

$$= \cfrac{x}{\alpha_1 + \cfrac{x^2}{\alpha_2 + \cfrac{x^2}{\alpha_3 + \cfrac{x^2}{\alpha_4 + \cfrac{x^2}{\alpha_5 + ...}}}}} \qquad (20.2)$$

Die Kettenbruchkoeffizienten α_i sind mit den normierten Ableitungswerten $F^{(2k+1)}$ von f(x) durch nachstehendes eindeutig umkehrbares infinites Rekursionsgleichungssystem verbunden.

Werteverlaufsgleichheit
in unterschiedlichen Definitionsbereichen und mit unterschiedlichen Konvergenzverhalten.

$$\alpha_1 = \beta_1(\mathrm{F}) = \frac{(-1)^{\mathrm{int}(\frac{1+1}{4})} = 1}{\mathrm{F}^{(1)}}$$

$$= \frac{(-1)^{\mathrm{int}(\frac{1+1}{4})+\mathrm{int}(\frac{1+2}{4})} = 1}{\mathrm{F}^{(1)}} = 1 \qquad (22.1)$$

$$\alpha_2 = \beta_2(\mathrm{F}) = \frac{(-1)^{\mathrm{int}(\frac{2+1}{4})} = 1}{\Delta_2}$$

$$\Delta_2 = \begin{vmatrix} \mathrm{F}^{(1)} & \mathrm{F}^{(3)} \\ 1 & 0 \end{vmatrix}$$

$$= \frac{(-1)^{\mathrm{int}(\frac{2+1}{4})+\mathrm{int}(\frac{2+2}{4})} = -1}{\mathrm{F}^{(3)}} \qquad (22.2)$$

$$\alpha_2^2\alpha_3 = \beta_3(\mathrm{F}) = \frac{(-1)^{\mathrm{int}(\frac{3+1}{4})} = -1}{\Delta_3}$$

$$\Delta_3 = \begin{vmatrix} \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} \\ 0 & \mathrm{F}^{(1)} & \mathrm{F}^{(3)} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{(-1)^{\text{int}(\frac{3+1}{4})+\text{int}(\frac{3+2}{4})} = 1}{\begin{vmatrix} F^{(1)} & F^{(3)} \\ F^{(3)} & F^{(5)} \end{vmatrix}} \tag{22.3}$$

$$\alpha_2^3\alpha_3^2\alpha_4 = \beta_4(F) = \frac{(-1)^{\text{int}(\frac{4+1}{4})} = -1}{\Delta_4 = \begin{vmatrix} F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} \\ 0 & F^{(1)} & F^{(3)} & F^{(5)} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}}$$

$$= \frac{(-1)^{\text{int}(\frac{4+1}{4})+\text{int}(\frac{4+2}{4})} = 1}{\begin{vmatrix} F^{(3)} & F^{(5)} \\ F^{(5)} & F^{(7)} \end{vmatrix}} \tag{22.4}$$

$$\alpha_2^4\alpha_3^3\alpha_4^2\alpha_5 = \beta_5(\mathrm{F}) = \frac{(-1)^{\mathrm{int}(\frac{5+1}{4})} = -1}{\Delta_5 = \begin{vmatrix} \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} & \mathrm{F}^{(9)} \\ 0 & \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} \\ 0 & 0 & \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}}$$

$$= \frac{(-1)^{\mathrm{int}(\frac{5+1}{4})+\mathrm{int}(\frac{5+2}{4})} = 1}{\begin{vmatrix} \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} \\ \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} \\ \mathrm{F}^{(5)} & \mathrm{F}^{(7)} & \mathrm{F}^{(9)} \end{vmatrix}} \qquad (22.5)$$

$$\alpha_2^5\alpha_3^4\alpha_4^3\alpha_5^2\alpha_6 = \beta_6(\mathrm{F})$$

$$= \frac{(-1)^{\mathrm{int}(\frac{6+1}{4})} = -1}{\Delta_6 = \begin{vmatrix} \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} & \mathrm{F}^{(9)} & \mathrm{F}^{(11)} \\ 0 & \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} & \mathrm{F}^{(9)} \\ 0 & 0 & \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix}}$$

$$= \frac{(-1)^{\mathrm{int}(\frac{6+1}{4})+\mathrm{int}(\frac{6+2}{4})} = -1}{\begin{vmatrix} \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & \mathrm{F}^{(7)} \\ \mathrm{F}^{(5)} & \mathrm{F}^{(7)} & \mathrm{F}^{(9)} \\ \mathrm{F}^{(7)} & \mathrm{F}^{(9)} & \mathrm{F}^{(11)} \end{vmatrix}} \qquad (22.6)$$

$$\alpha_2^6\alpha_3^5\alpha_4^4\alpha_5^3\alpha_6^2\alpha_7=\beta_7(\mathrm{F})$$

$$=\frac{(-1)^{\mathrm{int}(\frac{7+1}{4})}=1}{\Delta_7=\begin{vmatrix} F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} & F^{(13)} \\ 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} \\ 0 & 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} \\ 0 & 0 & 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{vmatrix}}$$

$$=\frac{(-1)^{\mathrm{int}(\frac{7+1}{4})+\mathrm{int}(\frac{7+2}{4})}=1}{\begin{vmatrix} F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} \\ F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} \\ F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} \\ F^{(7)} & F^{(9)} & F^{(11)} & F^{(13)} \end{vmatrix}} \qquad (22.7)$$

$$\alpha_2^7\alpha_3^6\alpha_4^5\alpha_5^4\alpha_6^3\alpha_7^2\alpha_8 = \beta_8(F) =$$

$$(-1)^{\text{int}(\frac{8+1}{4})} = 1$$

$$\Delta_8 = \begin{vmatrix} F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} & F^{(13)} & F^{(15)} \\ 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} & F^{(13)} \\ 0 & 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} \\ 0 & 0 & 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$$= \frac{(-1)^{\text{int}(\frac{8+1}{4})+\text{int}(\frac{8+2}{4})} = 1}{\begin{vmatrix} F^{(3)} & F^{(5)} & F^{(7)} & F^{(9)} \\ F^{(5)} & F^{(7)} & F^{(9)} & F^{(11)} \\ F^{(7)} & F^{(9)} & F^{(11)} & F^{(13)} \\ F^{(9)} & F^{(11)} & F^{(13)} & F^{(15)} \end{vmatrix}} \qquad (22.8)$$

Die Determinantenausdrücke $\beta_i(F)$ zeigen sowohl bezüglich Vorzeichen als auch Aufbau

der Determinanten $\Delta_k(F)$ einen Quantisierungseffekt.
Die Berechnung der α_i aus den $F^{(2k+1)}$ via Gl. (22) kann
** sowohl rekursiv aus den Gln (22.1,2,3,4,…)
** als auch durch „Mittelwertbildung“

$$\alpha_1 = \beta_1(F)\ ;\ \alpha_2 = \beta_2(F)$$

$$\alpha_i = \frac{\beta_{i-2}(F) * \beta_i(F)}{(\beta_{i-1}(F))^2}$$

$$= (-1)^{\mathrm{int}(\frac{i-1}{2})} \frac{(\Delta_{i-1}(F))^2}{\Delta_{i-2}(F) * \Delta_i(F)}\ ;\ i=3,4,5,... \qquad (23)$$

- a beautiful solution -,

erfolgen.

5.2 Rücktransformation $\alpha_i \rightarrow F^{(2k+1)}$

Die zugehörige Umkehrung: Berechnung der $F^{(2k+1)}$ aus den α_i via Gl. (22)
** erfolgt rekursiv

$$+F^{(1)} = \frac{1}{\alpha_1} = 1 \tag{24.1}$$

$$-F^{(3)} = \frac{1}{\alpha_2} - \Delta_2^*(F)\ ;\ \Delta_2^*(F) = \begin{vmatrix} F^{(1)} & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$$= \frac{1}{\alpha_2} \tag{24.2}$$

$$+F^{(5)} = \frac{1}{\alpha_2^2 \alpha_3} + \Delta_3^*(F)\ ;$$

$$\Delta_3^*(F) = \begin{vmatrix} F^{(1)} & F^{(3)} & 0 \\ 0 & F^{(1)} & F^{(3)} \\ 1 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} F^{(1)} & F^{(3)} \\ F^{(3)} & 0 \end{vmatrix}$$

$$= \frac{1}{\alpha_2^2 \alpha_3} + \frac{1}{\alpha_2^2} \tag{24.3}$$

$$-\mathrm{F}^{(7)} = \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4} + \alpha_2 \Delta_4^*(\mathrm{F}) ;$$

$$\Delta_4^*(\mathrm{F}) = \begin{vmatrix} \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} & 0 \\ 0 & \mathrm{F}^{(1)} & \mathrm{F}^{(3)} & \mathrm{F}^{(5)} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} \mathrm{F}^{(3)} & \mathrm{F}^{(5)} \\ \mathrm{F}^{(5)} & 0 \end{vmatrix}$$

$$= \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4} + \frac{1}{\alpha_2^3}\left(1 + \frac{1}{\alpha_3}\right)^2 \qquad (24.4)$$

$$+F^{(9)} = \frac{1}{\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5} + \alpha_2^2\alpha_3\Delta_5^*(F)\ ;$$

$$\Delta_5^*(F) = \begin{vmatrix} F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} & 0 \\ 0 & F^{(1)} & F^{(3)} & F^{(5)} & F^{(7)} \\ 0 & 0 & F^{(1)} & F^{(3)} & F^{(5)} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} F^{(1)} & F^{(3)} & F^{(5)} \\ F^{(3)} & F^{(5)} & F^{(7)} \\ F^{(5)} & F^{(7)} & 0 \end{vmatrix} \qquad (24.5)$$

.....

$$(-1)^{k-1}F^{(2\ \ k1)} = \frac{1}{(\alpha_2\alpha_3...\alpha_{k-1})^2\alpha_k}$$

$$+(-1)^{\mathrm{int}(\frac{k+1}{4})+1}\alpha_2^{k-3}\alpha_3^{k-4}...\alpha_{k-3}^2\alpha_{k-2}\Delta_k^*(F) \qquad (24.6)$$

** oder direkt:

$$+\,F^{(1)} = \frac{1}{\alpha_1} = 1$$

$$-\,F^{(3)} = \frac{1}{\alpha_2}$$

$$+\,F^{(5)} = \frac{1}{\alpha_2^2} + \frac{1}{\alpha_2^2 \alpha_3}$$

$$-\,F^{(7)} = \frac{1}{\alpha_2^3} + \frac{2}{\alpha_2^3 \alpha_3} + \frac{1}{\alpha_2^3 \alpha_3^2} + \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4}$$

$$= \frac{(1+\alpha_3)^2}{\alpha_2^3 \alpha_3^2} + \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4}$$

$$+\,F^{(9)} = \frac{(1+\alpha_3)^3}{\alpha_2^4 \alpha_3^3} + 2\frac{1+\alpha_3}{\alpha_2^3 \alpha_3^3 \alpha_4} + \frac{1}{\alpha_2^2 \alpha_3^3 \alpha_4^2}$$

$$+ \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5} \qquad (25.1\text{-}5)$$

Continue this sequence and find a recursion relation and a 2D representation for these 1D $\alpha_i \rightarrow F^{(2k+1)}$ formulas.

First approach

** Determinant representation of the infinite

simple continued fraction

$$f(x) = [0; \frac{\alpha_1}{x}, \frac{\alpha_2}{x}, \frac{\alpha_3}{x}, \frac{\alpha_4}{x}, \frac{\alpha_5}{x}, ...] \; ; \; \alpha_1 = 1$$

$$= \cfrac{1}{\frac{\alpha_1}{x} + \cfrac{1}{\frac{\alpha_2}{x} + \cfrac{1}{\frac{\alpha_3}{x} + \cfrac{1}{\frac{\alpha_4}{x} + \cfrac{1}{\frac{\alpha_5}{x} + ...}}}}} \qquad (20.1)$$

$$= \cfrac{x}{\alpha_1 + \cfrac{x^2}{\alpha_2 + \cfrac{x^2}{\alpha_3 + \cfrac{x^2}{\alpha_4 + \cfrac{x^2}{\alpha_5 + ...}}}}} \qquad (20.2)$$

$$
\begin{vmatrix}
\frac{x}{\alpha_1} & 0 & 0 & 0 & 0 & \rightarrow \\
0 & 1 & \frac{x}{\alpha_2} & 0 & 0 & \\
0 & -\frac{x}{\alpha_3} & 1 & \frac{x}{\alpha_3} & 0 & \\
0 & 0 & -\frac{x}{\alpha_4} & 1 & \frac{x}{\alpha_4} & \\
0 & 0 & 0 & -\frac{x}{\alpha_5} & 1 & \\
\downarrow & & & & & \searrow
\end{vmatrix}
$$

$$
\Rightarrow
\begin{vmatrix}
1 & \frac{x}{\alpha_1} & 0 & 0 & 0 & \rightarrow \\
-\frac{x}{\alpha_2} & 1 & \frac{x}{\alpha_2} & 0 & 0 & \\
0 & -\frac{x}{\alpha_3} & 1 & \frac{x}{\alpha_3} & 0 & \\
0 & 0 & -\frac{x}{\alpha_4} & 1 & \frac{x}{\alpha_4} & \\
0 & 0 & 0 & -\frac{x}{\alpha_5} & 1 & \\
\downarrow & & & & & \searrow
\end{vmatrix}
\qquad (20.3)
$$

** and transformation into an infinite rational function and finally into an infinite polynomial

$$\Rightarrow \frac{x}{\alpha_1}\frac{\sum_{i=0}^{\infty} b_i x^{2i}}{\sum_{i=0}^{\infty} a_i x^{2i}} \tag{20.4}$$

$$\Rightarrow -\frac{x}{\alpha_1}\begin{vmatrix} 0 & 1 & b_1 & b_2 & b_3 & \rightarrow \\ x^0 & 1 & a_1 & a_2 & a_3 & \\ x^2 & 0 & 1 & a_1 & a_2 & \\ x^4 & 0 & 0 & 1 & a_1 & \\ x^6 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & \searrow \end{vmatrix} \tag{20.5}$$

$$a_0 = b_0 = 1$$

$$= \sum_{k=0}^{\infty} F^{(2k+1)} x^{2k+1} \; ; F^{(1)} = 1 \tag{20.6}$$

Example 3
Representation of f(x) by an equation, Maclaurin series and infinite continued fraction.

$$f(x)=\frac{x}{\sqrt{1\pm x^2}}=\sum_{i=0}^{\infty}(\mp 1)^i\frac{\binom{2i}{i}}{2^{2i}}x^{2i+1}$$

$$=\sum_{i=0}^{\infty}F^{(2i+1)}x^{2i+1}\ ;\ |x|<1$$

$$F^{(1)}=1,\ F^{(3)}=\mp\frac{1}{2},\ F^{(5)}=\frac{3}{8},\ F^{(7)}=\mp\frac{5}{16},$$

$$F^{(9)}=\frac{35}{128},\ F^{(11)}=\mp\frac{63}{256},\ F^{(13)}=\frac{231}{1024},$$

$$F^{(15)}=\mp\frac{429}{2048},\ F^{(17)}=\frac{6435}{32768},$$

$$F^{(19)}=\mp\frac{12155}{2^{16}}=\mp\frac{5*11*13*17}{2^{16}},$$

$$F^{(21)}=\frac{46189}{2^{18}}=\frac{11*13*17*19}{2^{18}},$$

$$F^{(23)}=\mp\frac{88179}{2^{19}}=\mp\frac{3*7*13*17*19}{2^{19}},$$

$$F^{(25)}=\frac{676039}{2^{22}}=\frac{7*13*17*19*23}{2^{22}},$$

$$F^{(27)}=\mp\frac{1300075}{2^{23}}=\mp\frac{5^2*7*17*19*23}{2^{23}},\ \ldots$$

$$f(x) = \frac{x}{\sqrt{1 \pm x^2}} = [0; \frac{1}{x}, \frac{\pm 2}{x}, \frac{2}{x}, \frac{\pm 2}{x}, \frac{2}{x}, \frac{\pm 2}{x}, \ldots ;$$

$$\text{abs}(x) < 1$$

$$\alpha_1 = 1, \ \alpha_2 = \alpha_4 = \alpha_6 = \alpha_8 = \ldots = \pm 2,$$

$$\alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \ldots = 2$$

Die Berechnung bestätigt die Gln (22), (23) und (24).

Your problems

**** Werteverläufe (Graph) von**

1)

$$\sum_{i=0}^{\infty}(\mp 1)^{i}\frac{\binom{2i}{i}}{2^{2i}}x^{2i+1}=\sum_{i=0}^{\infty}F^{(2i+1)}x^{2i+1}$$

$$F^{(1)}=1,\ F^{(3)}=\mp\frac{1}{2},\ F^{(5)}=\frac{3}{8},\ F^{(7)}=\mp\frac{5}{16},$$

$$F^{(9)}=\frac{35}{128},\ F^{(11)}=\mp\frac{63}{256},\ F^{(13)}=\frac{231}{1024},$$

$$F^{(15)}=\mp\frac{429}{2048},\ F^{(17)}=\frac{6435}{32768},$$

$$F^{(19)}=\mp\frac{12155}{2^{16}}=\mp\frac{5*11*13*17}{2^{16}},$$

$$F^{(21)}=\frac{46189}{2^{18}}=\frac{11*13*17*19}{2^{18}},$$

$$F^{(23)}=\mp\frac{88179}{2^{19}}=\mp\frac{3*7*13*17*19}{2^{19}},$$

$$F^{(25)}=\frac{676039}{2^{22}}=\frac{7*13*17*19*23}{2^{22}},$$

$$F^{(27)}=\mp\frac{1300075}{2^{23}}=\mp\frac{5^{2}*7*17*19*23}{2^{23}},\ \ldots$$

fuer abs(x)<1 und fuer abs(x)>1.

2)

$$[0;\frac{1}{x},\frac{\pm 2}{x},\frac{2}{x},\frac{\pm 2}{x},\frac{2}{x},\frac{\pm 2}{x},..\ ]$$

fuer abs(x)<1 und fuer abs(x)>1.

** **Welche Funktionen stellen die infinite Ausdruecke 1) und 2) fuer abs(x)>1 dar?**

Example 4
Representation of f(x) by an equation, Maclaurin series and infinite continued fraction.

$$\left.\begin{matrix}\tanh x\\ \tan x\end{matrix}\right\} = x \mp \frac{1}{3}x^3 + \frac{2}{15}x^5 \mp \frac{17}{315}x^7 + \frac{62}{2835}x^9$$
$$\mp \frac{1382}{155925}x^{11} + \frac{21844}{6081075}x^{13}$$
$$\mp \frac{929569}{638512875}x^{15} + \frac{6404582}{10854718875}x^{17} \mp \ldots ;$$
$$-\frac{\pi}{2} < x < +\frac{\pi}{2} \qquad (26.1)$$

$$= [0; \frac{1}{y}, \frac{\pm 3}{y}, \frac{5}{y}, \frac{\pm 7}{y}, \frac{9}{y}, \frac{\pm 11}{y}, \frac{13}{y}, \frac{\pm 15}{y}, ...] \; ;$$

$$\begin{cases} y = x \\ y = 2\pi \text{fract}(\frac{x}{2\pi}) \quad [4] \end{cases} \qquad (26.2)$$

Die Berechnung bestätigt die Gln (22), (23) und (24).

What does the icf /tanx in equ. (26.2) represent for $|x| > 2\pi$?

Example 5
Representation of f(x) by an infinite continued fraction and Maclaurin series.

$$\text{primtanx} = [0;\frac{1}{x},\frac{-3}{x},\frac{5}{x},\frac{-7}{x},\frac{11}{x},\frac{-13}{x},\frac{17}{x},\frac{-19}{x},\frac{23}{x},\ldots$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{884}{40425}x^9$$

$$+ \frac{3227284}{364188825}x^{11} + \frac{111276341866}{3098700617512 5}x^{13}$$

$$+ \frac{4373949532079 5643}{3005643539267 9821125}x^{15}$$

$$+ \frac{6590544883814879952875 48}{1117562738953753047643813125}x^{17}$$

$$+ \frac{1117562738953753047643813125}{3455796213002675052094090340 0014}x^{19} + \ldots$$

(27)

Die Berechnung bestätigt die Gln (22), (23) und (24).

6. A determinant-based infinite continued fraction representation of f(x) = u(x)/v(x) for the special case

$u^{(2k)}(0) = v^{(2k+1)}(0) = 0$; k=0,1,2,3,…

$u^{(1)}(0) = v(0) = 1$

"In unserm Weinberg liegt ein Schatz, grabt nur danach!"
Gottfried August Bürger
The treasure is here the complete structural consistency of all determinants. Let's excavate more, see [1].

6.1 Transformation of the Maclaurin series of this f(x) into an infinite continued fraction

The Maclaurin series of

$$f(x) = \frac{u(x)}{v(x)} \quad \text{with } u^{(1)}(0) = v(0) = 1 \quad \text{and}$$

$$u^{(2k)}(0) = v^{(2k+1)}(0) = 0 \; ; \quad k=0,1,2,3,\ldots \qquad (28.1)$$

$$= -\begin{vmatrix} 0 & U^{(1)} & U^{(3)} & U^{(5)} & U^{(7)} & U^{(9)} & \rightarrow \\ x & 1 & V^{(2)} & V^{(4)} & V^{(6)} & V^{(8)} & \\ x^3 & 0 & 1 & V^{(2)} & V^{(4)} & V^{(6)} & \\ x^5 & 0 & 0 & 1 & V^{(2)} & V^{(4)} & \\ x^7 & 0 & 0 & 0 & 1 & V^{(2)} & \\ x^9 & 0 & 0 & 0 & 0 & 1 & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

(28.2)

can be reversibly transformed into an infinite continued fraction

$$f(x) = [0; \frac{\alpha_1}{x}, \frac{\alpha_2}{x}, \frac{\alpha_3}{x}, \frac{\alpha_4}{x}, \frac{\alpha_5}{x}, ...]$$

$$= \cfrac{1}{\cfrac{\alpha_1}{x} + \cfrac{1}{\cfrac{\alpha_2}{x} + \cfrac{1}{\cfrac{\alpha_3}{x} + \cfrac{1}{\cfrac{\alpha_4}{x} + \cfrac{1}{\cfrac{\alpha_5}{x} + ...}}}}} \qquad (20.1)$$

$$= \cfrac{x}{\alpha_1 + \cfrac{x^2}{\alpha_2 + \cfrac{x^2}{\alpha_3 + \cfrac{x^2}{\alpha_4 + \cfrac{x^2}{\alpha_5 + ...}}}}} \qquad (20.2)$$

Die Kettenbruchkoeffizienten α_i sind mit den normierten Ableitungswerten $U^{(2k+1)}$ und $V^{(2k)}$ von u(x) bzw. v(x) durch nachstehendes infinites Rekursionsgleichungssystem verbunden.

$$\alpha_1 = \beta_1(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{1+1}{4})} = 1}{\mathrm{U}^{(1)}} = 1$$

$$\alpha_2 = \beta_2(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{2+1}{4})} = 1}{\Delta_2(\mathrm{U},\mathrm{V})}$$

$$\Delta_2(\mathrm{U},\mathrm{V}) = \begin{vmatrix} \mathrm{U}^{(1)} & \mathrm{U}^{(3)} \\ \mathrm{V} & \mathrm{V}^{(2)} \end{vmatrix}$$

$$\alpha_2^2\alpha_3 = \beta_3(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{3+1}{4})} = -1}{\Delta_3(\mathrm{U},\mathrm{V})}$$

$$\Delta_3(\mathrm{U},\mathrm{V}) = \begin{vmatrix} \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} \\ 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} \\ \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} \end{vmatrix}$$

$$\alpha_2^3\alpha_3^2\alpha_4 = \beta_4(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{4+1}{4})}}{\Delta_4(\mathrm{U},\mathrm{V})} = -1$$

$$\Delta_4(\mathrm{U},\mathrm{V}) = \begin{vmatrix} \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} \\ 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} \\ \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \\ 0 & \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} \end{vmatrix}$$

$$\alpha_2^4\alpha_3^3\alpha_4^2\alpha_5 = \beta_5(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{5+1}{4})}}{\Delta_5(\mathrm{U},\mathrm{V})} = -1$$

$$\Delta_5(\mathrm{U},\mathrm{V}) = \begin{vmatrix} \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} & \mathrm{U}^{(9)} \\ 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} \\ 0 & 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} \\ \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \\ 0 & \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \end{vmatrix}$$

$$\alpha_2^5\alpha_3^4\alpha_4^3\alpha_5^2\alpha_6 = \beta_6(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{6+1}{4})} = -1}{\Delta_6(\mathrm{U},\mathrm{V})}$$

$$\Delta_6(\mathrm{U},\mathrm{V}) = \begin{vmatrix} \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} & \mathrm{U}^{(9)} & \mathrm{U}^{(11)} \\ 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} & \mathrm{U}^{(9)} \\ 0 & 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} \\ \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} \\ 0 & \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \\ 0 & 0 & \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \end{vmatrix}$$

$$\alpha_2^6\alpha_3^5\alpha_4^4\alpha_5^3\alpha_6^2\alpha_7 = \beta_7(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{7+1}{4})} = 1}{\Delta_7(\mathrm{U},\mathrm{V})}$$

$$\Delta_7(\mathrm{U},\mathrm{V}) = \begin{vmatrix} \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} & \mathrm{U}^{(9)} & \mathrm{U}^{(11)} & \mathrm{U}^{(13)} \\ 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} & \mathrm{U}^{(9)} & \mathrm{U}^{(11)} \\ 0 & 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} & \mathrm{U}^{(9)} \\ 0 & 0 & 0 & \mathrm{U}^{(1)} & \mathrm{U}^{(3)} & \mathrm{U}^{(5)} & \mathrm{U}^{(7)} \\ \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} & \mathrm{V}^{(12)} \\ 0 & \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} \\ 0 & 0 & \mathrm{V} & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \end{vmatrix}$$

(29.1-7)

Die Determinantenausdrücke $\beta_i(U,V)$ zeigen sowohl bezüglich Vorzeichen als auch Aufbau der Determinanten $\Delta_k(U,V)$ einen Quantisierungseffekt.

Die Berechnung der α_i aus den $U^{(2k)}$ und $V^{(2k-1)}$ via Gl. (22) kann

** sowohl rekursiv aus den Gln (22.1,2,3,4,…)

** als auch durch „Mittelwertbildung“

$$\alpha_1 = \beta_1(U,V)\ ;\ \alpha_2 = \beta_2(U,V)$$

$$\alpha_i = \frac{\beta_{i-2}(U,V) * \beta_i(U,V)}{(\beta_{i-1}(U,V))^2}$$

$$= (-1)^{\mathrm{int}(\frac{i-1}{2})} \frac{(\Delta_{i-1}(U,V))^2}{\Delta_{i-2}(U,V) * \Delta_i(U,V)}\ ;\ i=3,4,5,...$$

(30.1,2)

- a beautiful solution -,

erfolgen.

Example 6

$$u(x) = \sin x\ ;\ v(x) = \cos x\ ;\ f(x) = \tan x \qquad (8)$$

With equ. (8) we obtain a 2D infinite continued fraction representation for tanx

$$\tan x = [0; \frac{\alpha_1}{x}, \frac{\alpha_2}{x}, \frac{\alpha_3}{x}, \frac{\alpha_4}{x}, \frac{\alpha_5}{x}, ...] \; ; \; \alpha_1 = 1$$

$$= \cfrac{1}{\cfrac{\alpha_1}{x} + \cfrac{1}{\cfrac{\alpha_2}{x} + \cfrac{1}{\cfrac{\alpha_3}{x} + \cfrac{1}{\cfrac{\alpha_4}{x} + \cfrac{1}{\cfrac{\alpha_5}{x} + ...}}}}} \qquad (20.1)$$

$$= \cfrac{x}{\alpha_1 + \cfrac{x^2}{\alpha_2 + \cfrac{x^2}{\alpha_3 + \cfrac{x^2}{\alpha_4 + \cfrac{x^2}{\alpha_5 + ...}}}}} \qquad (20.2)$$

with

$$\alpha_1 = \frac{1}{\Delta_1(U,V)} = \frac{1}{U^{(1)}} = \frac{1}{\frac{1}{1!}} = 1 ;$$

$$\alpha_2 = \frac{1}{\Delta_2(U,V)} = \frac{1}{\begin{vmatrix} U^{(1)} & U^{(3)} \\ V & V^{(2)} \end{vmatrix}} = \frac{-1}{U^{(3)} - V^{(2)}}$$

$$= \frac{1}{\begin{vmatrix} \frac{1}{1!} & \frac{-1}{3!} \\ \frac{1}{0!} & \frac{-1}{2!} \end{vmatrix}} = \frac{-1}{\frac{-1}{3!} - \frac{-1}{2!}} = -3$$

$$\alpha_i = (-1)^{\text{int}(\frac{i-1}{2})} \frac{(\Delta_{i-1}(U,V))^2}{\Delta_{i-2}(U,V) * \Delta_i(U,V)} \; ; \; i=3,4,5,...$$

$$\alpha_3 = (-1)^{\text{int}(\frac{2}{2})} \frac{(\Delta_2(U,V))^2}{\Delta_1(U,V) * \Delta_3(U,V)}$$

$$= -\frac{\begin{vmatrix} U^{(1)} & U^{(3)} \\ V & V^{(2)} \end{vmatrix}^2}{U^{(1)} * \begin{vmatrix} U^{(1)} & U^{(3)} & U^{(5)} \\ 0 & U^{(1)} & U^{(3)} \\ V & V^{(2)} & V^{(4)} \end{vmatrix}}$$

$$= \frac{(U^{(3)} - V^{(2)})^2}{U^{(3)} * [(U^{(5)} - V^{(4)}) - U^{(3)}(U^{(3)} - V^{(2)})]}$$

$$= -\frac{\begin{vmatrix} \frac{1}{1!} & \frac{-1}{3!} \\ \frac{1}{0!} & \frac{-1}{2!} \end{vmatrix}^2}{\frac{1}{1!} * \begin{vmatrix} \frac{1}{1!} & \frac{-1}{3!} & \frac{1}{5!} \\ 0 & \frac{1}{1!} & \frac{-1}{3!} \\ \frac{1}{0!} & \frac{-1}{2!} & \frac{1}{4!} \end{vmatrix}} = \frac{-\frac{1}{9}}{-\frac{1}{45}} = 5$$

$$\alpha_4 = (-1)^{\text{int}(\frac{3}{2})} \frac{(\Delta_3(U,V))^2}{\Delta_2(U,V)*\Delta_4(U,V)}$$

$$= -\frac{\begin{vmatrix} \frac{1}{1!} & \frac{-1}{3!} & \frac{1}{5!} \\ 0 & \frac{1}{1!} & \frac{-1}{3!} \\ \frac{1}{0!} & \frac{-1}{2!} & \frac{1}{4!} \end{vmatrix}^2}{\begin{vmatrix} \frac{1}{1!} & \frac{-1}{3!} \\ \frac{1}{0!} & \frac{-1}{2!} \end{vmatrix} * \begin{vmatrix} \frac{1}{1!} & \frac{-1}{3!} & \frac{1}{5!} & \frac{-1}{7!} \\ 0 & \frac{1}{1!} & \frac{-1}{3!} & \frac{1}{5!} \\ \frac{1}{0!} & \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} \\ 0 & \frac{1}{0!} & \frac{-1}{2!} & \frac{1}{4!} \end{vmatrix}}$$

$$= \frac{-(\frac{-1}{45})^2}{(-\frac{1}{3})*(-\frac{1}{4725})} = -7$$

(31.1-4)

.....

Result

$$\tan x = [0; \frac{1}{x}, \frac{-3}{x}, \frac{5}{x}, \frac{-7}{x}, \frac{9}{x}, \frac{-11}{x}, \frac{13}{x}, \frac{-15}{x}, ...]$$

(31.5)

Zum Vergleich

$$\left.\begin{matrix} \tanh x \\ \tan x \end{matrix}\right\} = \begin{vmatrix} 0 & \frac{1}{1!} & \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{1}{9!} & \rightarrow \\ -x^1 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \frac{1}{8!} & \\ \mp x^3 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \\ -x^5 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \frac{1}{4!} & \\ \mp x^7 & 0 & 0 & 0 & \frac{1}{0!} & \frac{1}{2!} & \\ -x^9 & 0 & 0 & 0 & 0 & \frac{1}{0!} & \\ \downarrow & & & & & & \searrow \end{vmatrix}$$

$$= x \mp \frac{1}{3}x^3 + \frac{2}{15}x^5 \mp \frac{17}{315}x^7 + \frac{62}{2835}x^9$$
$$\mp \frac{1382}{155925}x^{11} + \frac{21844}{6081075}x^{13}$$
$$\mp \frac{929569}{638512875}x^{15} + \frac{6404582}{10854718875}x^{17} \mp \ldots ;$$

$$-\frac{\pi}{2} < x < +\frac{\pi}{2}$$

$$= [0; \frac{1}{y}, \frac{\pm 3}{y}, \frac{5}{y}, \frac{\pm 7}{y}, \frac{9}{y}, \frac{\pm 11}{y}, \frac{13}{y}, \frac{\pm 15}{y}, \ldots] ;$$

$$y = a*\mathrm{fract}(\frac{x}{a}) ; \begin{cases} a = \infty \\ a = 2\pi \end{cases}$$

(32)

6.2 Rücktransformation $\alpha_i \rightarrow U^{(2k-1)}, V^{(2k)}$

Special case u(x) = x

$U^{(1)} = V = 1$

$U = U^{(2)} = U^{(3)} = U^{(4)} = U^{(5)} = U^{(6)} = \ldots =$

$V^{(1)} = V^{(3)} = V^{(5)} = V^{(7)} = V^{(9)} = V^{(11)} = \ldots = 0$

(33.1)

this means

$$f(x) = [0; \frac{\alpha_1}{x}, \frac{\alpha_2}{x}, \frac{\alpha_3}{x}, \frac{\alpha_4}{x}, \frac{\alpha_5}{x}, ...]$$

$$= \cfrac{1}{\cfrac{\alpha_1}{x} + \cfrac{1}{\cfrac{\alpha_2}{x} + \cfrac{1}{\cfrac{\alpha_3}{x} + \cfrac{1}{\cfrac{\alpha_4}{x} + \cfrac{1}{\cfrac{\alpha_5}{x} + ...}}}}}$$

$$= \cfrac{x}{\alpha_1 + \cfrac{x^2}{\alpha_2 + \cfrac{x^2}{\alpha_3 + \cfrac{x^2}{\alpha_4 + \cfrac{x^2}{\alpha_5 + ...}}}}}$$

$$= \frac{x}{v(x)} = \frac{x}{V + \sum_{k=1}^{\infty} V^{(2k)} x^{(2k)}} \; ; \; \alpha_1 = V = 1$$

(33.2)

Die Kettenbruchkoeffizienten α_i sind dann mit den normierten Ableitungswerten $V^{(2k)}$

von v(x) durch nachstehendes infinites Rekursionsgleichungssystem verbunden.

$$\alpha_1 = \beta_1(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{1+1}{4})} = 1}{1} = 1$$

$$\alpha_2 = \beta_2(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{2+1}{4})} = 1}{\Delta_2(\mathrm{U},\mathrm{V})}$$

$$\Delta_2(\mathrm{U},\mathrm{V}) = \begin{vmatrix} 1 & 0 \\ 1 & \mathrm{V}^{(2)} \end{vmatrix} = \mathrm{V}^{(2)}$$

$$\alpha_2^2\alpha_3 = \beta_3(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{3+1}{4})} = -1}{\Delta_3(\mathrm{U},\mathrm{V})}$$

$$\Delta_3(\mathrm{U},\mathrm{V}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} \end{vmatrix} = \mathrm{V}^{(4)}$$

$$\alpha_2^3\alpha_3^2\alpha_4 = \beta_4(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{4+1}{4})} = -1}{\Delta_4(\mathrm{U},\mathrm{V})}$$

$$\Delta_4(\mathrm{U},\mathrm{V}) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \\ 0 & 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} \end{vmatrix}$$

$$= -\begin{vmatrix} \mathrm{V}^{(2)} & \mathrm{V}^{(4)} \\ \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \end{vmatrix}$$

$$\alpha_2^4\alpha_3^3\alpha_4^2\alpha_5 = \beta_5(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{5+1}{4})} = -1}{\Delta_5(\mathrm{U},\mathrm{V})}$$

$$\Delta_5(\mathrm{U},\mathrm{V}) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \\ 0 & 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \end{vmatrix}$$

$$= -\begin{vmatrix} \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \\ \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \end{vmatrix}$$

$$\alpha_2^5\alpha_3^4\alpha_4^3\alpha_5^2\alpha_6 = \beta_6(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{6+1}{4})} = -1}{\Delta_6(\mathrm{U},\mathrm{V})}$$

$$\Delta_6(\mathrm{U},\mathrm{V}) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} \\ 0 & 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \\ 0 & 0 & 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \end{vmatrix}$$

$$= -\begin{vmatrix} \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \\ \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \\ \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} \end{vmatrix}$$

$$\alpha_2^6\alpha_3^5\alpha_4^4\alpha_5^3\alpha_6^2\alpha_7 = \beta_7(\mathrm{U},\mathrm{V}) = \frac{(-1)^{\mathrm{int}(\frac{7+1}{4})} = 1}{\Delta_7(\mathrm{U},\mathrm{V})}$$

$$\Delta_7(\mathrm{U},\mathrm{V}) =$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} & \mathrm{V}^{(12)} \\ 0 & 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} \\ 0 & 0 & 1 & \mathrm{V}^{(2)} & \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \end{vmatrix}$$

$$= -\begin{vmatrix} \mathrm{V}^{(4)} & \mathrm{V}^{(6)} & \mathrm{V}^{(8)} \\ \mathrm{V}^{(6)} & \mathrm{V}^{(8)} & \mathrm{V}^{(10)} \\ \mathrm{V}^{(8)} & \mathrm{V}^{(10)} & \mathrm{V}^{(12)} \end{vmatrix}$$

(34.1-7)

Nach Umformung erhält man daraus für die Linearprodukte der α_i

$$\gamma_1 = \prod_{i=1}^{1} \alpha_i = +1$$

$$\gamma_2 = \prod_{i=1}^{2} \alpha_i = + \frac{1}{\left| V^{(2)} \right|}$$

$$\gamma_3 = \prod_{i=1}^{3} \alpha_i = - \frac{\left| V^{(2)} \right|}{\left| V^{(4)} \right|}$$

$$\gamma_4 = \prod_{i=1}^{4} \alpha_i = - \frac{\left| V^{(4)} \right|}{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}}$$

$$\gamma_5 = \prod_{i=1}^{5} \alpha_i = + \frac{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}}$$

$$\gamma_6 = \prod_{i=1}^{6} \alpha_i = + \frac{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}}{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}}$$

$$\gamma_7 = \prod_{i=1}^{7} \alpha_i = - \frac{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(8)} & V^{(10)} & V^{(12)} \end{vmatrix}}$$

$$\gamma_8 = \prod_{i=1}^{8} \alpha_i = - \frac{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(8)} & V^{(10)} & V^{(12)} \end{vmatrix}}{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(4)} & V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(6)} & V^{(8)} & V^{(10)} & V^{(12)} \\ V^{(8)} & V^{(10)} & V^{(12)} & V^{(14)} \end{vmatrix}}$$

$$\gamma_9 = \prod_{i=1}^{9} \alpha_i = + \frac{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(4)} & V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(6)} & V^{(8)} & V^{(10)} & V^{(12)} \\ V^{(8)} & V^{(10)} & V^{(12)} & V^{(14)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(6)} & V^{(8)} & V^{(10)} & V^{(12)} \\ V^{(8)} & V^{(10)} & V^{(12)} & V^{(14)} \\ V^{(10)} & V^{(12)} & V^{(14)} & V^{(16)} \end{vmatrix}}$$

.....

(35.1-9)

aus denen die α_i rekursiv leicht berechnet werden können:

$$\alpha_k = \frac{\gamma_k}{\gamma_{k-1}} \; ; \; k=2,3,4,... \qquad (36)$$

Die zugehörige Umkehrung: Berechnung der $V^{(2k)}$ aus den α_i via Gl. (??)

** erfolgt rekursiv

$$+\,\mathrm{V} = \frac{1}{\alpha_1} = 1$$

$$+\,\mathrm{V}^{(2)} = \frac{1}{\alpha_2}$$

$$-\,\mathrm{V}^{(4)} = \frac{1}{\alpha_2^2 \alpha_3}$$

$$+\,\mathrm{V}^{(6)} = \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4} - \frac{\begin{vmatrix} \mathrm{V}^{(2)} & \mathrm{V}^{(4)} \\ \mathrm{V}^{(4)} & 0 \end{vmatrix}}{\left|\mathrm{V}^{(2)}\right|}$$

$$-\,\mathrm{V}^{(8)} = \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5} + \frac{\begin{vmatrix} \mathrm{V}^{(4)} & \mathrm{V}^{(6)} \\ \mathrm{V}^{(6)} & 0 \end{vmatrix}}{\left|\mathrm{V}^{(4)}\right|}$$

$$+ V^{(10)} = \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5^2 \alpha_6} - \frac{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & 0 \end{vmatrix}}{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}}$$

$$- V^{(12)} = \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5^2 \alpha_6^2 \alpha_7} + \frac{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(8)} & V^{(10)} & 0 \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}}$$

$$+\,V^{(14)} = \frac{1}{\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2\alpha_6^2\alpha_7^2\alpha_8}$$

$$\frac{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(4)} & V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(6)} & V^{(8)} & V^{(10)} & V^{(12)} \\ V^{(8)} & V^{(10)} & V^{(12)} & 0 \end{vmatrix}}{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}}$$

$$+\,V^{(16)} = \frac{1}{\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2\alpha_6^2\alpha_7^2\alpha_8^2\alpha_9}$$

$$+\frac{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(6)} & V^{(8)} & V^{(10)} & V^{(12)} \\ V^{(8)} & V^{(10)} & V^{(12)} & V^{(14)} \\ V^{(10)} & V^{(12)} & V^{(14)} & 0 \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(4)} & V^{(6)} & V^{(10)} \\ V^{(8)} & V^{(10)} & V^{(12)} \end{vmatrix}}$$

(37.1-8)

** oder direkt

$$+\ V = \frac{1}{\alpha_1} = 1$$

$$+\ V^{(2)} = \frac{1}{\alpha_2}$$

$$-\ V^{(4)} = \frac{1}{\alpha_2^2 \alpha_3}$$

$$+\ V^{(6)} = \frac{1}{\alpha_2^3 \alpha_3^2} + \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4} = \frac{1}{\alpha_2^2 \alpha_3^2}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_4}\right)$$

$$-\ V^{(8)} = \frac{1}{\alpha_2^4 \alpha_3^3} + \frac{2}{\alpha_2^3 \alpha_3^3 \alpha_4} + \frac{1}{\alpha_2^2 \alpha_3^3 \alpha_4^2} + \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5}$$

$$= \frac{1}{\alpha_2^2 \alpha_3^3}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_4}\right)^2 + \frac{1}{\alpha_2^2 \alpha_3^2 \alpha_4^2 \alpha_5}$$

$$= \frac{1}{\alpha_2^2 \alpha_3^2}\left[\frac{1}{\alpha_3}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_4}\right)^2 + \frac{1}{\alpha_4^2 \alpha_5}\right]$$

$$+\ V^{(10)} = \frac{1}{\alpha_2^2 \alpha_3^2}\left[\frac{1}{\alpha_3^2}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_4}\right)^3 + \frac{2}{\alpha_3 \alpha_4^2 \alpha_5}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_4}\right) + \frac{1}{\alpha_4^2 \alpha_5^2}\left(\frac{1}{\alpha_4} + \frac{1}{\alpha_6}\right)\right]$$

$$-\ V^{(12)} = \frac{1}{\alpha_2^2 \alpha_3^2}\left[\frac{1}{\alpha_3^3}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_4}\right)^4 + \ldots\right.$$

Continue this sequence and find a recursion relation and a 2D representation for these 1D $\alpha_i \rightarrow V^{(2k)}$ formulas.

Example 9

$V^{(2k)} \rightarrow \alpha_i$

$u(x) = x,\ v(x) = \cos x$

$U = 0,\ U^{(1)} = 1,\ U^{(2)} = U^{(4)} = U^{(6)} = \ldots = 0$

$V = 1,\ V^{(2k)} = \dfrac{(-1)^k}{(2k)!},\ V^{(2k+1)} = 0;\ k = 1,2,3,4,\ldots$

$$\gamma_1 = \prod_{i=1}^{1} \alpha_i = 1$$

$$\gamma_2 = \prod_{i=1}^{2} \alpha_i = \frac{1}{V^{(2)}} = \frac{1}{\frac{-1}{2!}} = -2$$

$$\gamma_3 = \prod_{i=1}^{3} \alpha_i = -\frac{V^{(2)}}{V^{(4)}} = -\frac{\frac{-1}{2!}}{\frac{1}{4!}} = 12$$

$$\gamma_4 = \prod_{i=1}^{4} \alpha_i = -\frac{\left|V^{(4)}\right|}{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}} = -\frac{\left|\frac{1}{4!}\right|}{\begin{vmatrix} \frac{-1}{2!} & \frac{1}{4!} \\ \frac{1}{4!} & \frac{-1}{6!} \end{vmatrix}} = 40$$

$$\gamma_5 = \prod_{i=1}^{5} \alpha_i = + \frac{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{-1}{2!} & \frac{1}{4!} \\ \frac{1}{4!} & \frac{-1}{6!} \end{vmatrix}}{\begin{vmatrix} \frac{1}{4!} & \frac{-1}{6!} \\ \frac{-1}{6!} & \frac{1}{8!} \end{vmatrix}} = \frac{15120}{13}$$

$$\gamma_6 = \prod_{i=1}^{6} \alpha_i = + \frac{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}}{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{4!} & \frac{-1}{6!} \\ \frac{-1}{6!} & \frac{1}{8!} \end{vmatrix}}{\begin{vmatrix} \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} \\ \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} \\ \frac{-1}{6!} & \frac{1}{8!} & \frac{-1}{10!} \end{vmatrix}}$$

$$= -\frac{131040}{59} = -\frac{2^5 * 3^2 * 5 * 7 * 13}{59}$$

$$\gamma_7 = \prod_{i=1}^{7} \alpha_i = -\frac{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(8)} & V^{(10)} & V^{(12)} \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} \frac{-1}{2!} & \frac{1}{4!} & \frac{-1}{6!} \\ \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} \\ \frac{-1}{6!} & \frac{1}{8!} & \frac{-1}{10!} \end{vmatrix}}{\begin{vmatrix} \frac{1}{4!} & \frac{-1}{6!} & \frac{1}{8!} \\ \frac{-1}{6!} & \frac{1}{8!} & \frac{-1}{10!} \\ \frac{1}{8!} & \frac{-1}{10!} & \frac{1}{12!} \end{vmatrix}}$$

$$= \frac{2^6 * 3^3 * 5 * 7 * 11 * 59}{127}$$

$$\gamma_8 = \prod_{i=1}^{8} \alpha_i = -\frac{2^7 * 3^4 * 5^2 * 11 * 13 * 127}{29 * 1229}$$

$$\gamma_9 = \prod_{i=1}^{9} \alpha_i = \frac{2^8 * 3^3 * 5^2 * 7^2 * 11 * 29 * 1229}{11321}$$

$$\alpha_1 = 1,\ \alpha_2 = -2,\ \alpha_3 = -2 * 3,\ \alpha_4 = \frac{2 * 5}{3},$$

$$\alpha_5 = \frac{2 * 3^3 * 7}{13},\ \alpha_6 = -\frac{2^4 * 5 * 13^2}{3 * 59},$$

$$\alpha_7 = -\frac{2 * 3 * 11 * 59^2}{13 * 127},\ \alpha_8 = \frac{2 * 3 * 5 * 13 * 127^2}{7 * 29 * 59 * 1229},$$

$$\alpha_9 = \frac{2 * 7^2 * 29^2 * 1229^2}{3 * 13 * 127 * 11321},\ \ldots$$

Example 10

$V^{(2k)} \longrightarrow \alpha_i$

$u(x) = x,\ v(x) = \dfrac{\sin x}{x}$

$U = 0,\ U^{(1)} = 1,\ U^{(2)} = U^{(4)} = U^{(6)} = \ldots = 0$

$$V = 1,\ V^{(2k)} = \frac{(-1)^k}{(2k+1)!},\ V^{(2k+1)} = 0;$$

$$k = 1,2,3,4,\ldots$$

$$\gamma_1 = \prod_{i=1}^{1} \alpha_i = 1$$

$$\gamma_2 = \prod_{i=1}^{2} \alpha_i = \frac{1}{V^{(2)}} = \frac{1}{\frac{-1}{3!}} = -6$$

$$\gamma_3 = \prod_{i=1}^{3} \alpha_i = -\frac{V^{(2)}}{V^{(4)}} = -\frac{\frac{-1}{3!}}{\frac{1}{5!}} = 20$$

$$\gamma_4 = \prod_{i=1}^{4} \alpha_i = -\frac{\left|V^{(4)}\right|}{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}} = -\frac{\left|\frac{1}{5!}\right|}{\begin{vmatrix} \frac{-1}{3!} & \frac{1}{5!} \\ \frac{1}{5!} & \frac{-1}{7!} \end{vmatrix}} = \frac{2520}{11}$$

$$\gamma_5 = \prod_{i=1}^{5} \alpha_i = + \frac{\begin{vmatrix} V^{(2)} & V^{(4)} \\ V^{(4)} & V^{(6)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{-1}{3!} & \frac{1}{5!} \\ \frac{1}{5!} & \frac{-1}{7!} \end{vmatrix}}{\begin{vmatrix} \frac{1}{5!} & \frac{-1}{7!} \\ \frac{-1}{7!} & \frac{1}{9!} \end{vmatrix}} = \frac{11088}{5}$$

$$\gamma_6 = \prod_{i=1}^{6} \alpha_i = + \frac{\begin{vmatrix} V^{(4)} & V^{(6)} \\ V^{(6)} & V^{(8)} \end{vmatrix}}{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{1}{5!} & \frac{-1}{7!} \\ \frac{-1}{7!} & \frac{1}{9!} \end{vmatrix}}{\begin{vmatrix} \frac{-1}{3!} & \frac{1}{5!} & \frac{-1}{7!} \\ \frac{1}{5!} & \frac{-1}{7!} & \frac{1}{9!} \\ \frac{-1}{7!} & \frac{1}{9!} & \frac{-1}{11!} \end{vmatrix}}$$

$$= -\frac{2^5 * 3^4 * 5^3 * 7}{11^2}$$

$$\gamma_7 = \prod_{i=1}^{7} \alpha_i = -\frac{\begin{vmatrix} V^{(2)} & V^{(4)} & V^{(6)} \\ V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \end{vmatrix}}{\begin{vmatrix} V^{(4)} & V^{(6)} & V^{(8)} \\ V^{(6)} & V^{(8)} & V^{(10)} \\ V^{(8)} & V^{(10)} & V^{(12)} \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} \frac{-1}{3!} & \frac{1}{5!} & \frac{-1}{7!} \\ \frac{1}{5!} & \frac{-1}{7!} & \frac{1}{9!} \\ \frac{-1}{7!} & \frac{1}{9!} & \frac{-1}{11!} \end{vmatrix}}{\begin{vmatrix} \frac{1}{5!} & \frac{-1}{7!} & \frac{1}{9!} \\ \frac{-1}{7!} & \frac{1}{9!} & \frac{-1}{11!} \\ \frac{1}{9!} & \frac{-1}{11!} & \frac{1}{13!} \end{vmatrix}}$$

$$= \frac{2^6 * 3^3 * 5 * 11^4 * 13}{43 * 61}$$

$$\gamma_8 = \prod_{i=1}^{8} \alpha_i = \frac{2^7 * 3^4 * 5^2 * 7^2 * 11 * 13 * 43 * 61}{17^2 * 6121}$$

$$\gamma_9 = \prod_{i=1}^{9} \alpha_i = \frac{2^8 * 3^2 * 5 * 7 * 11 * 13 * 17^3 * 6121}{1029037}$$

$$\alpha_1 = 1,\ \alpha_2 = -2 * 3,\ \alpha_3 = \frac{-2 * 5}{3},\ \alpha_4 = \frac{2 * 3^2 * 7}{11},$$

$$\alpha_5 = \frac{2 * 11^2}{5^2},\ \alpha_6 = -\frac{2 * 3^2 * 5^4}{11^3},$$

$$\alpha_7 = -\frac{2 * 11^6 * 13}{3 * 5^2 * 7 * 43 * 61},$$

$$\alpha_8 = \frac{2 * 3 * 5 * 7^2 * 43^2 * 61^2}{11^3 * 17^2 * 6121},$$

$$\alpha_9 = \frac{2^8 * 3^2 * 5 * 7 * 11 * 13 * 17^3 * 6121}{1029037},\ \ldots$$

Example 11

$\alpha_i \rightarrow V^{(2k)}$

$\alpha_1=1, \alpha_2=3, \alpha_3=5, \alpha_4=7, \alpha_5=9, \alpha_6=11, \ldots \rightarrow$

$$V=1=\frac{2^0}{0!}*\ =\frac{2^0}{0!}1*B_0$$

$$V^{(2)}=\frac{1}{3}=\frac{2^2}{2!}*\frac{1}{3}=\frac{2^2}{2!}*B_2$$

$$V^{(4)}=\frac{-1}{3^2*5}=\frac{2^4}{4!}*\frac{-1}{30}=\frac{2^4}{4!}*B_4$$

$$V^{(6)}=\frac{2}{3^3*5*7}=\frac{2^6}{6!}*\frac{1}{42}=\frac{2^6}{6!}*B_6$$

$$V^{(8)}=\frac{-1}{3^3*5^2*7}=\frac{2^8}{8!}*\frac{-1}{30}=\frac{2^8}{8!}*B_8$$

$$V^{(10)}=\frac{2}{3^5*5*7*11}=\frac{2^{10}}{10!}*\frac{5}{66}=\frac{2^{10}}{10!}*B_{10}$$

$$V^{(12)}=\frac{-2*691}{3^6*5^3*7^2*11*13}=\frac{2^{12}}{12!}*\frac{-691}{2730}=\frac{2^{12}}{12!}*B_{12}$$

691– Bernoulli marker!

……

$$V^{(2k)}=\frac{2^{2k}}{(2k)!}*B_{2k}\ ;\ k=0,1,2,3,4,...$$

B_{2k} – Bernoulli numbers

$$v(x) = \sum_{k=0}^{\infty} \frac{2^{2k} * B_{2k}}{(2k)!}\ x^{2k}\ = x*\text{cthx}\ ;\ 0 \le \text{abs}(x) \le \pi$$

$$v(\sqrt{-\sqrt{-1}}*x) = \text{re}[v(\sqrt{-\sqrt{-1}}*x)]$$

$$+ \sqrt{-1}*\text{im}[v(\sqrt{-\sqrt{-1}}*x)]$$

$$= 1 - \frac{2^4 * B_4}{4!} x^4 + \frac{2^8 * B_8}{8!} x^8 - \frac{2^{12} * B_{12}}{12!} x^{12} \pm \ldots$$

$$+\sqrt{-1}*\left(- \frac{2^2 * B_2}{2!} x^2 + \frac{2^6 * B_6}{6!} x^6 - \frac{2^{10} * B_{10}}{10!} x^{10} \mp \ldots\right)$$

$$\text{re}[v(\sqrt{-\sqrt{-1}}*x)] = \text{re}(v) =$$

$$1 - \frac{2^4 * B_4}{4!} x^4 + \frac{2^8 * B_8}{8!} x^8 - \frac{2^{12} * B_{12}}{12!} x^{12} \pm \ldots$$

$$\text{im}[v(\sqrt{-\sqrt{-1}}*x)] = \text{im}(v) =$$

$$- \frac{2^2 * B_2}{2!} x^2 + \frac{2^6 * B_6}{6!} x^6 - \frac{2^{10} * B_{10}}{10!} x^{10} \mp \ldots$$

Example 12

$\alpha_i \rightarrow V^{(2k)}$

$\alpha_1=1$, $\alpha_2=3$, $\alpha_3=5$, $\alpha_4=7$, $\alpha_5=11$, $\alpha_6=13$, $\alpha_7=17$,

$\alpha_i=prim_i \rightarrow$

$+V=1$

$$+V^{(2)}=\frac{1}{3}$$

$$-V^{(4)}=\frac{1}{3^2*5}$$

$$+V^{(6)}=\frac{2}{3^3*5*7}$$

$$-V^{(8)}=\frac{229}{3^4*5^2*7^2*11}$$

$$+V^{(10)}=\frac{2^2*9287}{3^3*7^3*11^2*13}$$

$$-V^{(12)}=\frac{2*3593*243403}{3^6*5^3*7^4*11^3*13^2*17}$$

$$+V^{(14)}=\frac{2*59*307*38803*120569}{3^7*5^3*7^5*11^4*13^3*17^2*19}$$

……

References

[1] http://www.spiegel.de/wissenschaft/mensch/mathematiker-faltings-kritisiert-mittelmass-an-deutschen-hochschulen-a-924142.html
„Man kann nicht sagen, das ist ein wichtiges Problem, und das will ich jetzt lösen. Ich muss auch mathematische Methoden haben, mit denen das gelingt. An den meisten großen Problemen haben sich schon viele kluge Leute versucht - und man sollte nicht unbedingt damit rechnen, dass man klüger ist als die. Die einzige Chance ist, ein mathematisches Werkzeug zur Hand zu nehmen, das die anderen noch nicht hatten. Wenn ich ein solches neues Werkzeug hätte, würde ich das Problem lösen.“

[2] http://mathworld.wolfram.com/BernoulliNumber.html
[3] http://en.wikipedia.org/wiki/Bernoulli_number
[4] http://mathworld.wolfram.com/EulerNumber.html
[5] http://en.wikipedia.org/wiki/Euler_number
[6] http://mathworld.wolfram.com/FractionalPart.html

01.06.2014
Unfinished. Find the keystone.