Unit 1249

Creative Formula Manipulation, Sand Box Part 1

Creative formula manipulation is based in a large part on mastering the dialectic between freedom and determination in arithmetic:

- Freedom:
  - * Commutative and associative laws of addition and multiplication are the basis for a free arrangement (and re-arrangement) and free structured bracketing of operands.
  - * Free and creative use of the two directions of the identity laws and its derivatives

$$a+0=a$$
$$a-a=0$$
$$a*1=a$$
$$a/a=1$$
$$a*0=0$$
$$(-1)*(-1)=1$$

Determination:

- * Distributive law of multiplication with respect to addition
- * Priority law + < * < ^ .

Example 1
Power sums and Pascal's triangle

Now is $\sum_{i=1}^{n} i^4 \rightarrow \sum_{i=1}^{n} i^m = p_m(n)$; m,n ∈ ℕ on the stage.

We start light-footed with a simple identity

$4^k = 4^k$; $k \in N_0$

Now we insert the additive identity 0 (invariant)

$4^k = 4^k + 0 + 0 + 0 - 0$.

Next step: extension of 0 by an opposite pair

$4^k = 4^k + (3^k-3^k) + (2^k-2^k) + (1^k-1^k) - 0^k$:

* "creatio ex nihilo" = creation out of nothing

* Go backwards to go forwards.
* A chess-player plans always some moves in advance.

Last step: reordering:

$$4^k = (4^k\text{-}3^k) + (3^k\text{-}2^k) + (2^k\text{-}1^k) + (1^k\text{-}0^k) = \sum_{i=1}^{4} i^k - (i\text{-}1)^k \quad .$$

Now generalization 4 → n:

Using the binomial formula, we obtain

$$n^k = \sum_{i=1}^{n} [i^k - (i-1)^k] = \sum_{i=1}^{n} [i^k - \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-j} i^j) - i^k] =$$

$$= \sum_{i=1}^{n} [\sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-1-j} i^j)] = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-1-j} [\sum_{i=1}^{n} i^j])$$

and with

$$\sum_{i=1}^{n} i^j = f(j,n)$$

$$n^k = \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k-1-j} f(j,n) \; ; \; k=1,2,3,4,5,.. \;\; ,$$

detailed for k=1(1)5

$$+n^1 = \binom{1}{0} f(0,n)$$

$$-n^2 = \binom{2}{0} f(0,n) - \binom{2}{1} f(1,n)$$

$$+n^3 = \binom{3}{0} f(0,n) - \binom{3}{1} f(1,n) + \binom{3}{2} f(2,n)$$

$$-n^4 = \binom{4}{0} f(0,n) - \binom{4}{1} f(1,n) + \binom{4}{2} f(2,n) - \binom{4}{3} f(3,n)$$

$$+n^5 = \binom{5}{0} f(0,n) - \binom{5}{1} f(1,n) + \binom{5}{2} f(2,n) - \binom{5}{3} f(3,n) + \binom{5}{4} f(4,n) \quad ,$$

in matrix representation

$$\begin{pmatrix} +n^1 \\ -n^2 \\ +n^3 \\ -n^4 \\ +n^5 \end{pmatrix} = \begin{pmatrix} +\binom{1}{0} & 0 & 0 & 0 & 0 \\ +\binom{2}{0} & -\binom{2}{1} & 0 & 0 & 0 \\ +\binom{3}{0} & -\binom{3}{1} & +\binom{3}{2} & 0 & 0 \\ +\binom{4}{0} & -\binom{4}{1} & +\binom{4}{2} & -\binom{4}{3} & 0 \\ +\binom{5}{0} & -\binom{5}{1} & +\binom{5}{2} & -\binom{5}{3} & +\binom{5}{4} \end{pmatrix} \begin{pmatrix} f(0,n) \\ f(1,n) \\ f(2,n) \\ f(3,n) \\ f(4,n) \end{pmatrix} .$$

Then f(4,n) can be found using Cramer's rule
http://mathworld.wolfram.com/CramersRule.html
after slight rearrangement as

$$f(4,n)=\sum_{i=1}^{n} i^4=\frac{n}{(4+1)!}\begin{vmatrix} +n^0 & \binom{1}{0} & 0 & 0 & 0 \\ -n^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +n^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -n^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +n^{3+1} & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}$$

determinant of (4+1)th order

$$=\frac{n}{5!}\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}$$

Pascal's triangle

determinant of 5th order

and after normalization

$$\frac{f(4,n)}{f(0,n)} = \frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0 = n} = \frac{\begin{vmatrix} +n^0 & \binom{1}{0} & 0 & 0 & 0 \\ -n^1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +n^2 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -n^3 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +n^4 & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}}{\begin{vmatrix} +1 & \binom{1}{0} & 0 & 0 & 0 \\ -1 & \binom{2}{0} & \binom{2}{1} & 0 & 0 \\ +1 & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & 0 \\ -1 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ +1 & \binom{4+1}{0} & \binom{4+1}{1} & \binom{4+1}{2} & \binom{4+1}{3} \end{vmatrix}} = (4+1)!$$

determinants of (4+1)th order

$$= \frac{1}{5!}\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix} = \frac{\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} +1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ +1 & 1 & 3 & 3 & 0 \\ -1 & 1 & 4 & 6 & 4 \\ +1 & 1 & 5 & 10 & 10 \end{vmatrix}} \qquad \text{(c)}$$

determinants of 5th order

$$= \sum_{j=0}^{4} A(4,j)\, n^j = p_4(n)$$

$$= \begin{vmatrix} A(4,0) & -1 & 0 & 0 & 0 \\ A(4,1) & n & -1 & 0 & 0 \\ A(4,2) & 0 & n & -1 & 0 \\ A(4,3) & 0 & 0 & n & -1 \\ A(4,4) & 0 & 0 & 0 & n \end{vmatrix} \qquad \text{(d)}$$

determinant of 5th order

Now generalization 4 → n.

Proof by mathematical induction.

http://en.wikipedia.org/wiki/Mathematical_induction

Comment:

The 2D determinant Pascal pattern representation (c)

$$\frac{\sum_{i=1}^{n} i^4}{\sum_{i=1}^{n} i^0 = n} = \frac{\begin{vmatrix} +n^0 & 1 & 0 & 0 & 0 \\ -n^1 & 1 & 2 & 0 & 0 \\ +n^2 & 1 & 3 & 3 & 0 \\ -n^3 & 1 & 4 & 6 & 4 \\ +n^4 & 1 & 5 & 10 & 10 \end{vmatrix}}{\begin{vmatrix} +1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ +1 & 1 & 3 & 3 & 0 \\ -1 & 1 & 4 & 6 & 4 \\ +1 & 1 & 5 & 10 & 10 \end{vmatrix}} = 5!$$

Pascal's triangle determinants
of 5th order

is the **"king size" solution** of the given problem:
* beautiful, simple, visually convincing,
* **extensible, portable!**

Oh, more!

$$\frac{\sum_{i=1}^{n} i^6}{n} = \frac{\begin{vmatrix} (-n)^0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-n)^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-n)^2 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-n)^3 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-n)^4 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-n)^5 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-n)^6 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{\begin{vmatrix} (-1)^0 & 1 & 0 & 0 & 0 & 0 & 0 \\ (-1)^1 & 1 & 2 & 0 & 0 & 0 & 0 \\ (-1)^2 & 1 & 3 & 3 & 0 & 0 & 0 \\ (-1)^3 & 1 & 4 & 6 & 4 & 0 & 0 \\ (-1)^4 & 1 & 5 & 10 & 10 & 5 & 0 \\ (-1)^5 & 1 & 6 & 15 & 20 & 15 & 6 \\ (-1)^6 & 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$= p_6(n) = \frac{1}{42}(6n^6+21n^5+21n^4+11n^2+1)$$

Plot $p_6(x)$; $x \in N$ and $p_6(z)$; $z \in R$.

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{6}}{1}=\frac{\begin{vmatrix} n \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ n^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ n^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ n^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ n^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ n^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ n^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}{\begin{vmatrix} 1 \bmod 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1^{1} & -1 & 2 & 0 & 0 & 0 & 0 \\ 1^{2} & 1 & -2 & 2 & 0 & 0 & 0 \\ 1^{3} & -1 & 3 & -3 & 2 & 0 & 0 \\ 1^{4} & 1 & -4 & 6 & -4 & 2 & 0 \\ 1^{5} & -1 & 5 & -10 & 10 & -5 & 2 \\ 1^{6} & 1 & -6 & 15 & -20 & 15 & -6 \end{vmatrix}}$$

(6+1) by (6+1) determinants

$$D(k=7,n)=\begin{vmatrix} \rho^0=1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \rho^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \rho^2 & 0 & 3 & 1 & 0 & 0 & 0 \\ \rho^3 & 0 & 1 & 6 & 1 & 0 & 0 \\ \rho^4 & 0 & 0 & 5 & 10 & 1 & 0 \\ \rho^5 & 0 & 0 & 1 & 15 & 15 & 1 \\ \rho^6 & 0 & 0 & 0 & 7 & 35 & 21 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n}(-1)^{n-i}i^{2k}}{\sum_{i=1}^{n}(-1)^{n-i}i^{2}}=\frac{D(k,n)}{D(k,1)}\;;\;\rho=n(n+1)$$

$$\sum_{i=1}^{n}(-1)^{n-i}i^{2}=\sum_{i=1}^{n}i^{1}=\frac{1}{2}n(n+1)$$

$$\lim_{k\to\infty}\binom{2k+2}{2}\frac{D(k,0)}{D(k+1,0)}=\frac{\pi^2}{2}$$

$$B_{2k}=(-1)^{k-1}\frac{D(k,0)}{2(2^{2k}-1)}$$ Bernoulli numbers

$$D(k=7,n)=\begin{vmatrix} \rho^0=1 & 2 & 0 & 0 & 0 & 0 & 0 \\ \rho^1 & 1 & 3 & 0 & 0 & 0 & 0 \\ \rho^2 & 0 & 4 & 4 & 0 & 0 & 0 \\ \rho^3 & 0 & 1 & 10 & 5 & 0 & 0 \\ \rho^4 & 0 & 0 & 6 & 20 & 6 & 0 \\ \rho^5 & 0 & 0 & 1 & 21 & 35 & 7 \\ \rho^6 & 0 & 0 & 0 & 8 & 56 & 56 \end{vmatrix}$$

7 by 7 determinant

$$\frac{\sum_{i=1}^{n} i^{2k+1}}{\sum_{i=1}^{n} i^3}=\frac{D(k,n)}{D(k,1)}\ ;\ \rho=n(n+1)$$

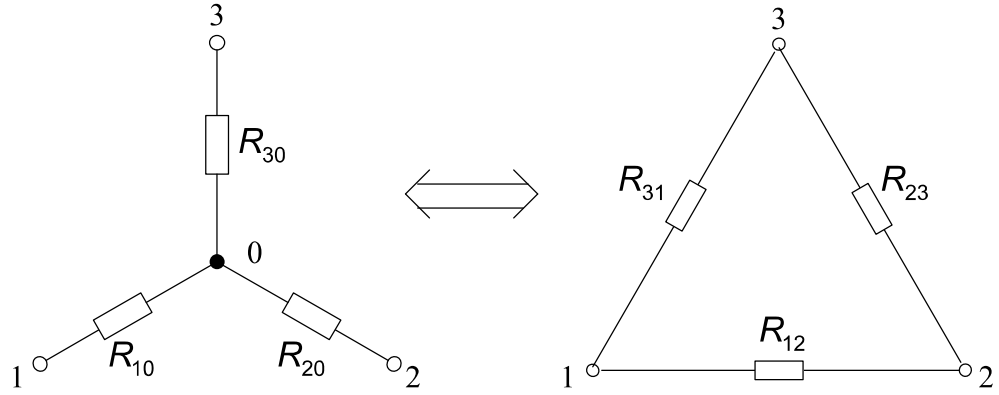
$$\sum_{i=1}^{n} i^3=\left(\sum_{i=1}^{n} i^1\right)^2=\frac{\rho^2}{4}$$

$$\lim_{k\to\infty}\binom{(2k+1)+3}{3}\frac{D(k,0)}{D(k+1,0)}=\frac{4}{3}\pi^2$$

Example 2
The bidirectional star-delta conversion, a standard problem of electrical system analysis

Example 1 showed: creative and efficient formula manipulation is an important part of mathematical description and analysis of signals. In the following we discuss from that point of view a standard problem of electrical system analysis.

Find a simplest way (minimal number of intermediate steps) to derive an as simple as possible equation system for the bidirectional star-delta conversion:



Star circuit with the Resistors R_{10}, R_{20} and R_{30}

Delta circuit with the Resistors R_{12}, R_{23} and R_{31}

Both circuits are equivalent regarding the terminals 1, 2, and 3. We assume the terminals are open, the circuits aren't part of a larger electrical network.

Solution

1. The equivalent resistance between the terminals 1 and 2 of the star circuit

$$Z_{star12} = R_{10} + R_{20} \qquad (1)$$

and of the delta circuit

$$Z_{delta12} = R_{12} \text{ parallel } (R_{23} + R_{31}) = \frac{R_{12}(R_{23} + R_{31})}{R_{12} + R_{23} + R_{31}} \qquad (2)$$

are equal. So we have

$$R_{10} + R_{20} = \frac{R_{12}R_{23} + R_{12}R_{31}}{R_{12} + R_{23} + R_{31}} \; . \qquad (3)$$

2. In the same way we obtain from

$$Z_{star23} = Z_{delta23} \text{ and } Z_{star31} = Z_{delta31} \; , \qquad (4)$$

or simpler by cyclic permutation in equ. (3) (the given problem is symmetrical)

$$R_{20} + R_{30} = \frac{R_{12}R_{23} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \; , \qquad (5)$$

and

$$R_{10} + R_{30} = \frac{R_{12}R_{31} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \; . \qquad (6)$$

3. Step 1 und 2 were very simple, the more difficult problem is now to separate R_{10}, R_{20}, and R_{30} on the left sides of equations that right side are expressions only in R_{12}, R_{23}, and R_{31}.
To find an efficient way to solve problems of this kind, it's often very useful to colorencode the relevant variables (one-to-one mapping), in our simple case R_{10}, R_{20}, and R_{30} in the equ. (3), (5), and (6):

$$R_{10} + R_{20} = \frac{R_{12}R_{23} + R_{12}R_{31}}{R_{12} + R_{23} + R_{31}} \qquad (3)$$

$$R_{20} + R_{30} = \frac{R_{12}R_{23} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \qquad (5)$$

$$R_{10} + R_{30} = \frac{R_{12}R_{31} + R_{23}R_{31}}{R_{12} + R_{23} + R_{31}} \; . \qquad (6)$$

So, it is obvious that we can separate $\begin{matrix} R_{10} \\ R_{20} \\ R_{30} \end{matrix}$ by the operations

equ. (3) +equ. (6) – equ. (5) $\qquad R_{10} = \frac{R_{12}R_{31}}{R_{12} + R_{23} + R_{31}}$

equ. (3) +equ. (5) – equ. (6) , obtaining $R_{20} = \frac{R_{12}R_{23}}{R_{12}+R_{23}+R_{31}}$

equ. (5) +equ. (6) – equ. (3) $R_{30} = \frac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}}$.

(7, 8, 9)

More:

The practice to color a problem and the way to its solution on your "inner display" (from black-white to color display!), in your mind, this is a very useful method to improve your formula handling skill, it's a creativity tool.

References:

- http://www.answers.com/synesthesia
- http://www.answers.com/Daniel%20Tammet

"Daniel Paul Tammet (b. January 31, 1979; London)
Experiencing numbers as colors or sensations is a well-documented form of synesthesia, but the detail and specificity of Tammet's mental imagery of numbers is unique. In his mind, he says, each number up to 10,000 has its own unique shape and feel, that he can "see" results of calculations as landscapes, and that he can "sense" whether a number is prime or composite. He has described his visual image of 289 as particularly ugly, 333 as particularly attractive, and pi as beautiful. Tammet not only verbally describes these visions, but also creates artwork, particularly watercolor paintings, such as his painting of Pi."

D. Tammet. Born on a Blue Day: Inside the Extraordinary Mind of an Autistic Savant.
Free Press 2007

4. The equs (7), (8), and (9) describe the delta to star conversion: $R_{12}, R_{23}, R_{31} \rightarrow R_{10}, R_{20}, R_{30}$, but this is only the half of the solution. To find the other part, the star to delta conversion $R_{10}, R_{20}, R_{30} \rightarrow R_{12}, R_{23}, R_{31}$, we had to solve the equ. system (7), (8), and (9) for R_{12}, R_{23}, and R_{31}. Terrible! However, there's a surprisingly simple way. Look at equ. (7):

$$R_{10} = \frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}} \qquad (7)$$

If we multiply both sides of this equ. by R_{23} than we obtain

$$R_{10}R_{23} = \frac{R_{12}R_{23}R_{31}}{R_{12}+R_{23}+R_{31}} \quad , \tag{10}$$

on the right side a symmetrical expression in R_{12}, R_{23}, and R_{31} (extension of an equation with the goal of symmetrization) invariant to cyclic permutation, the invariant of the delta circuit

$$R^2_{delta} = \frac{R_{12}R_{23}R_{31}}{R_{12}+R_{23}+R_{31}} \tag{11.1}$$

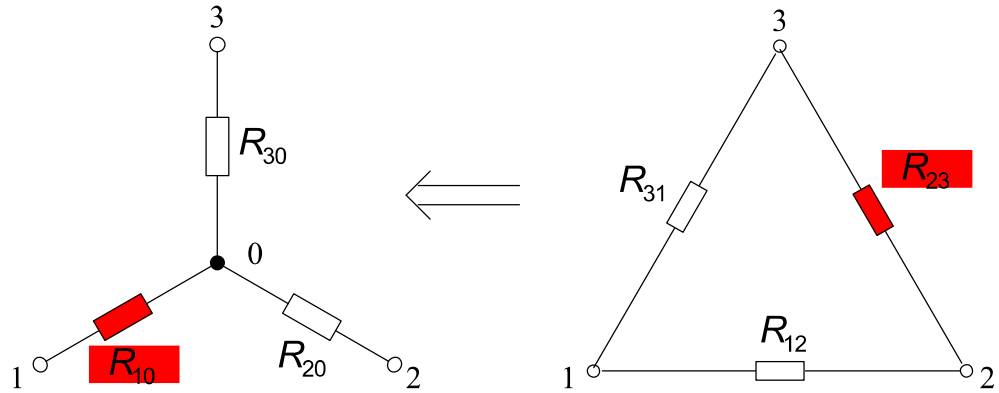
$$= \frac{1}{\frac{1}{R_{12}R_{23}}+\frac{1}{R_{12}R_{31}}+\frac{1}{R_{23}R_{31}}} \tag{11.2}$$

$$\frac{1}{R^2_{delta}} = \frac{1}{R_{12}R_{23}}+\frac{1}{R_{12}R_{31}}+\frac{1}{R_{23}R_{31}} \tag{11.3}$$

On the left side of equ. (10) we have the "opposite pair" $R_{10}R_{23}$

regarding the invariant R^2_{delta}:

$$R_{10}R_{23} = R^2_{delta} \tag{12.1}$$



5. In the same way we obtain the formula system

$$R_{10}R_{23} = R^2_{delta} \quad \rightarrow \quad R_{23} = \frac{R^2_{delta}}{R_{10}} \tag{12.1,2}$$

$$R_{20}R_{31} = R^2_{delta} \quad \rightarrow \quad R_{31} = \frac{R^2_{delta}}{R_{20}} \tag{13.1,2}$$

$$R_{30}R_{12} = R^2_{delta} \quad \rightarrow \quad R_{12} = \frac{R^2_{delta}}{R_{30}} \tag{14.1,2))}$$

The substitution of equs (12.2), (13.2), and (14.2) in equ. (11.2) results in

$$\frac{1}{R_{delta}^2} = \frac{1}{R_{12}R_{23}} + \frac{1}{R_{12}R_{31}} + \frac{1}{R_{23}R_{31}}$$

$$= \frac{1}{\frac{R_{delta}^2}{R_{30}}\frac{R_{delta}^2}{R_{10}}} + \frac{1}{\frac{R_{delta}^2}{R_{30}}\frac{R_{delta}^2}{R_{20}}} + \frac{1}{\frac{R_{delta}^2}{R_{10}}\frac{R_{delta}^2}{R_{20}}}$$

$$= \frac{1}{R_{delta}^4}(R_{10}R_{20} + R_{10}R_{30} + R_{20}R_{30})$$

and finally

$$R_{delta}^2 = R_{10}R_{20} + R_{10}R_{30} + R_{20}R_{30} = R_{star}^2 \quad !!! \qquad (15)$$

The values of the invariants R_{delta}^2 and R_{star}^2 are equal:

$$R_{delta}^2 = R_{star}^2 = R^2 \ . \qquad (16)$$

More:

To find and to handle invariants and opposite pairs (interdependence of opposites, unity in duality) in an efficient formula design process is a further useful creativity tool

References:

http://www.answers.com/topic/invariant-mathematics

http://www.answers.com/topic/invariant-physics

http://www.answers.com/Yin%20and%20Yang

By the way, the opposite-pair strategy is also a very effective approach to learn and to memorize vocabulary and sentences in a foreign language:

the day	the night
large	small
full	empty
even number	odd number
to transmit an encoded message	to encode a received message
to switch on	to switch off
the capacitor/capacitance	the inductor/inductance

Great opportunity and great danger are two sides of the same coin.

6. Operation scheduling (step by step) in advance

A masterly chess-player plans always his moves in advance.

... the power of accurately calculating moves in advance has no greater place in chess than, perhaps, skilful calculation has in mathematics." .

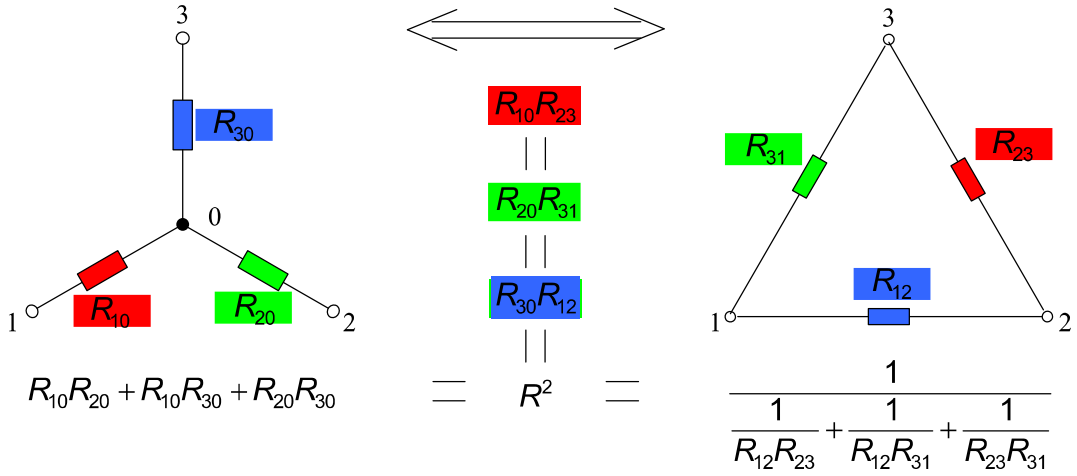
More:

http://www.chesscircle.net/forums/general-chess-forum/10897-how-many-moves-ahead.htm

http://seattlepi.nwsource.com/archives/1988/8801130570.asp

7. Result

Bidirectional star-delta conversion



Find more applications for our toolbox:



in mathematics, physics, electrical engineering, and computer science.

Part 2

Scoutics

How we can find and use a low situated simple pass way through an extremely high problem solving (esp. calculation) mountains? Why do such passes ("wormholes") exist?

Example 1

Real polynomials

$$p_n(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{j=1}^{n} (x - x_j)$$

have n (real or complex) zeros x_j.

Finite general formula expressions in radicals exist for this zeros only for n=2, 3, and 4.

Hovewer, in special ("wormhole") cases (integer coefficients a_i or multiple zeros) exist formula expressions involving only the arithmetic operations. This is our battle field!

Problem 1

It is known that a given polynomial of 3rd order in x

$p_3(x) = x^3 + a_2x^2 + a_1x + a_0$ (S.1)

has the product structure

$p_3(x) = (x - x_1)(x - x_2)^2$; x_1 unequal x_2 (S.2)

(one simple zero, two double zeros).

Example for equ. (S1):

$p_3(x) = x^3 + x^2 - 8x - 12 = 0$ (S.3.1)

has the roots

$x_1 = 3$; $x_2 = -2$ (S.3.2)

and the product structure

$p_3(x) = (x - 3)(x + 2)^2$. (S.3.3)

Find a simple procedure for calculation of the zeros x_1 and x_2 using only the four arithmetic operations addition, subtraction, multiplication and division, **no root operations** (solution of the general equation of the third order $p_3(x) = x^3 + a_2x^2 + a_1x + a_0$ in radicals: Tartaglia/Cardano formula)!

1. Special case: integer polynomial

$p_3(x)$ with (S.2) is an integer polynomial: x, all a_i , and all x_j are integers.

Solution

** Roots

Set $v = -\operatorname{sgn}(a_1 * a_0) = -\operatorname{sgn}(p_3'(0) * p_3(0))$ (S.4)

Set $g = \gcd(\operatorname{abs}(a_1), \operatorname{abs}(a_0)) = \gcd(\operatorname{abs}(p_3'(0)), \operatorname{abs}(p_3(0)))$ (S.5)

If g odd then set r = 1, else set r = 2 (S.6)

Set $x_2 = v*g/r$ (S.7)

Set $x_1 = v*(\operatorname{abs}(a_2) - 2*\operatorname{abs}(x_2))$ (S.8)

For the above example we have

$a_2 = 1;\ a_1 = -8;\ a_0 = -12 \rightarrow v = -1;\ g = 4;\ r = 2 \rightarrow x_1 = 3;\ x_2 = -2$

** Basic polynomial

$$q_2(p_3(x) = (x-x_1)(x-x_2)^2) = (x-x_1)(x-x_2)$$

$$= x^2 + v[\frac{g}{r} - \operatorname{abs}(a_2)]x + \frac{g}{r}[\operatorname{abs}(a_2) - 2\frac{g}{r}] \qquad \text{(S.9)}$$

Explain the solution process (S.4-8) step by step.

2. General case: real polynomial

$p_3(x)$ with (S.2) is a real polynomial: x, all a_i , and all x_j are reals.

Solution

** Roots

$$x_1 = -\frac{\begin{vmatrix} 0 & 2 & a_2 \\ a_2 & 2a_1 & 3a_0 \\ 3 & 2a_2 & a_1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ a_2 & 2a_1 & 3a_0 \\ 3 & 2a_2 & a_1 \end{vmatrix}};\ x_2 = \frac{\begin{vmatrix} 0 & 1 & 0 \\ a_2 & 2a_1 & 3a_0 \\ 3 & 2a_2 & a_1 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ a_2 & 2a_1 & 3a_0 \\ 3 & 2a_2 & a_1 \end{vmatrix}} \qquad \text{(S10.1;2)}$$

** Basis polynomial

$$q_2(p_3(x)=(x-x_1)(x-x_2)^2)=(x-x_1)(x-x_2)\ ;\ x_1\neq x_2$$

$$=x^2+\frac{\begin{vmatrix}0 & 1 & a_2\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}}{\begin{vmatrix}0 & 0 & 1\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}}x^1+\frac{\begin{vmatrix}1 & a_2 & a_1\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}}{\begin{vmatrix}0 & 0 & 1\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}}x^0$$

$$=\frac{\begin{vmatrix}x^0 & x^1+a_2x^0 & x^2+a_2x^1+a_1x^0\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}}{\begin{vmatrix}0 & 0 & 1\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}} \qquad \text{(S.11)}$$

** Remainder polynomial

$$r_1(p_3(x))=ggT(p_3(x),p_3'(x))=\frac{p_3(x)}{q_2(p_3(x))}=(x-x_2)$$

$$=x^1+\frac{\begin{vmatrix}a_2 & 3a_0\\ 3 & 1a_1\end{vmatrix}}{\begin{vmatrix}a_2 & 2a_1\\ 3 & 2a_2\end{vmatrix}}x^0$$

$$=\frac{\begin{vmatrix}0 & -x^0 & x\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}}{\begin{vmatrix}0 & 0 & 1\\ a_2 & 2a_1 & 3a_0\\ 3 & 2a_2 & a_1\end{vmatrix}} \qquad \text{(S.12.1;2)}$$

Explain the solution process (S.10.1;2), (S.11), and (S.12) step by step.

** Questions:

1. Necessary and sufficient condition that $p_3(x)$ (S.1) has the product structure (S.2)?
   Answer:

$$\begin{vmatrix} a_2 & 2a_1 \\ 3 & 2a_2 \end{vmatrix} \neq 0 \; ; \begin{vmatrix} a_2 & 2a_1 & 3a_0 & 0 \\ 3 & 2a_2 & a_1 & 0 \\ 0 & a_2 & 2a_1 & 3a_0 \\ 0 & 3 & 2a_2 & a_1 \end{vmatrix} = 0 \qquad \text{(S.13)}$$

2. Applications of the above results in science and engineering?
3. Interpretation of the above results in the case "all zeros of $p_3(x)$ are in pairs distinct".

Extension 1

Extend the solutions (S.4) – (S.8) and (S10.1;2) – (S.12) of the equs (S.1), (S.2) on the

Problem 2

It is known that the given real polynomial of n^{th} order in x

$$p_n(x) = \sum_{i=0}^{n} a_i x^i \; ; a_n = 1 \qquad \text{(S.14)}$$

has the product structure

$$p_n(x) = (x - x_1)^{k1}(x - x_2)^{k2} \; ; k_1 \neq k_2 \; ; k_1 + k_2 = n \, . \qquad \text{(S.15)}$$

Find determinant expressions for the zeros $x_{1;2}$, the basic and the remainder polynomials of $p_n(x)$.

Solution

** Zeros

$$x_{1;2} = \frac{1}{k_{1;2} - k_{2;1}} \frac{\begin{vmatrix} 0 & k_{2;1} & (k_{2;1}\text{-}1)a_{n-1} \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n\text{-}1)a_{n-1} & (n\text{-}2)a_{n-2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n\text{-}1)a_{n-1} & (n\text{-}2)a_{n-2} \end{vmatrix}} \qquad \text{(S.16)}$$

** Basis polynomial

$$q_2(p_n(x)=(x-x_1)^{k1}(x-x_2)^{k2}) = (x-x_1)(x-x_2)\ ;\ x_1 \neq x_2$$

$$= x^2 + \frac{\begin{vmatrix} 0 & 1 & a_{n-1} \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} \end{vmatrix}} x^1 + \frac{\begin{vmatrix} 1 & a_{n-1} & a_{n-2} \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} \end{vmatrix}} x^0$$

$$= \frac{\begin{vmatrix} x^0 & x^1+a_{n-1}x^0 & x^2+a_{n-1}x^1+a_{n-2}x^0 \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ a_{n-1} & 2a_{n-2} & 3a_{n-3} \\ n & (n-1)a_{n-1} & (n-2)a_{n-2} \end{vmatrix}} \qquad \text{(S.17)}$$

** Remainder polynomial?

** Necessary and sufficient condition that $p_n(x)$ (S.14) has the product structure (S.15)?

Extension 2

Extend the solutions (S.4) – (S.8) and (S.10.1;2) – (S.12) of the equs (S.1), (S.2) on the

Problem 3

It is known that the given polynomial of 6th order in x

$p_6(x) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ (S.18)

has the product representation

$p_6(x) = (x - x_1)^1(x - x_2)^2(x - x_3)^3$; all x_j in pairs distinct (S.19)

(one simple zero, 2 double zeros and 3 threefold zeros).

Find a simple procedure for the calculation of the zeros x_1, x_2 and x_3 (integer and real case), basic, and remainder polynomials using only the four arithmetic operations addition, subtraction multiplication and division, **no root operations** (you know: solution of the general equation of sixth order (S.10) in radicals is impossible)**!**

1. Special case: integer polynomial

$p_6(x)$ with (S.13) is an integer polynomial: x, all a_i , and all x_j are integers.

Solution: your turn! Firstly, check the web.

** Roots

** Basic polynomial

2. General case: real polynomial

$p_6(x)$ with (S.13) is a real polynomial: x, all a_i , and all x_j are reals.

Solution

** Roots, find determinant representation

** Basic polynomial in determinant representation

$$q_3(p_6(x) = (x-x_1)^1(x-x_2)^2(x-x_3)^3) = (x-x_1)(x-x_2)(x-x_3)$$

$$= x^3 + \frac{\begin{vmatrix} 0 & 0 & 0 & 1 & a_5 \\ a_5 & 2a_4 & 3a_3 & 4a_2 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 \\ 0 & a_5 & 2a_4 & 3a_3 & 4a_2 \\ 0 & 6 & 5a_5 & 4a_4 & 3a_5 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ a_5 & 2a_4 & 3a_3 & 4a_2 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 \\ 0 & a_5 & 2a_4 & 3a_3 & 4a_2 \\ 0 & 6 & 5a_5 & 4a_4 & 3a_5 \end{vmatrix}} x^2$$

$$+ \frac{\begin{vmatrix} 0 & 0 & 1 & a_5 & a_4 \\ a_5 & 2a_4 & 3a_3 & 4a_2 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 \\ 0 & a_5 & 2a_4 & 3a_3 & 4a_2 \\ 0 & 6 & 5a_5 & 4a_4 & 3a_5 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ a_5 & 2a_4 & 3a_3 & 4a_2 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 \\ 0 & a_5 & 2a_4 & 3a_3 & 4a_2 \\ 0 & 6 & 5a_5 & 4a_4 & 3a_5 \end{vmatrix}} x + \frac{\begin{vmatrix} 0 & 1 & a_5 & a_4 & a_3 \\ a_5 & 2a_4 & 3a_3 & 4a_2 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 \\ 0 & a_5 & 2a_4 & 3a_3 & 4a_2 \\ 0 & 6 & 5a_5 & 4a_4 & 3a_5 \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ a_5 & 2a_4 & 3a_3 & 4a_2 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 \\ 0 & a_5 & 2a_4 & 3a_3 & 4a_2 \\ 0 & 6 & 5a_5 & 4a_4 & 3a_5 \end{vmatrix}} \qquad \text{(S.20)}$$

Test

$x_1=1$; $x_2=2$; $x_3=-1$

$\rightarrow p_6(x) = x^6 - 2x^5 - 4x^4 + 6x^3 + 7x^2 - 4x - 4$

$= (x-1)^1(x-2)^2(x+1)^3$

$q_3(p_6(x)) = (x-1)(x-2)(x+1) = x^3 - 2x^2 - x + 2$

** Remainder polynomial in determinant representation

$$r_3(p_6(x)) = ggT(p_6(x), p_6'(x)) = \frac{p_6(x)}{q_3(p_6(x)}$$

$$= (x - x_1)^{1-1=0}(x - x_2)^{2-1=1}(x - x_3)^{3-1=2}$$

$$= x^3 + \frac{\begin{vmatrix} a_5 & 2a_4 & 3a_3 & 5a_1 \\ 6 & 5a_5 & 4a_4 & 2a_2 \\ 0 & a_5 & 2a_4 & 4a_2 \\ 0 & 6 & 5a_5 & 3a_3 \end{vmatrix}}{\begin{vmatrix} a_5 & 2a_4 & 3a_3 & 4a_2 \\ 6 & 5a_5 & 4a_4 & 3a_3 \\ 0 & a_5 & 2a_4 & 3a_3 \\ 0 & 6 & 5a_5 & 4a_4 \end{vmatrix}} x^2$$

$$+ \frac{\begin{vmatrix} a_5 & 2a_4 & 3a_3 & 6a_0 \\ 6 & 5a_5 & 4a_4 & a_1 \\ 0 & a_5 & 2a_4 & 5a_1 \\ 0 & 6 & 5a_5 & 2a_2 \end{vmatrix}}{\begin{vmatrix} a_5 & 2a_4 & 3a_3 & 4a_2 \\ 6 & 5a_5 & 4a_4 & 3a_3 \\ 0 & a_5 & 2a_4 & 3a_3 \\ 0 & 6 & 5a_5 & 4a_4 \end{vmatrix}} x$$

$$+ \frac{\begin{vmatrix} a_5 & 2a_4 & 3a_3 & 0 \\ 6 & 5a_5 & 4a_4 & 0 \\ 0 & a_5 & 2a_4 & 6a_0 \\ 0 & 6 & 5a_5 & a_1 \end{vmatrix}}{\begin{vmatrix} a_5 & 2a_4 & 3a_3 & 4a_2 \\ 6 & 5a_5 & 4a_4 & 3a_3 \\ 0 & a_5 & 2a_4 & 3a_3 \\ 0 & 6 & 5a_5 & 4a_4 \end{vmatrix}} \qquad \text{(S.21)}$$

To find x_2 and x_3 go to (S.1), (S.2), and (S.10.1;2).

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