Unit 100, corrected and extended

Real polynomials and their properties, determinant representation

Start

$$p_n(x) = \sum_{i=0}^{n} a_i x^i;\ a_0 = 1$$

$$= \begin{vmatrix} 1 & -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_n \\ x^1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ x^2 & 0 & 1 & 0 & 0 & \cdots & 0 \\ x^3 & 0 & 0 & 1 & 0 & \cdots & 0 \\ x^4 & 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ x^n & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\frac{1}{p_n(x)} = \frac{1}{\sum_{i=0}^{n} a_i x^i};\ a_0 = 1$$

$$= \begin{vmatrix} 1 & a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ x^1 & 1 & a_1 & a_2 & a_3 & \ddots & \vdots \\ x^2 & 0 & 1 & a_1 & a_2 & \ddots & a_4 \\ x^3 & 0 & 0 & 1 & a_1 & \ddots & a_3 \\ x^4 & 0 & 0 & 0 & 1 & \ddots & a_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & a_1 \\ x^n & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\frac{\sum_{i=0}^{\infty} b_i x^i}{\sum_{i=0}^{\infty} a_i x^i} = -\begin{vmatrix} 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ x^0 & 1 & a_1 & a_2 & a_3 & a_4 & \cdots \\ x^1 & 0 & 1 & a_1 & a_2 & a_3 & \cdots \\ x^2 & 0 & 0 & 1 & a_1 & a_2 & \cdots \\ x^3 & 0 & 0 & 0 & 1 & a_1 & \cdots \\ x^4 & 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix};\ a_0 = 1$$

$$\frac{\sum_{i=0}^{\infty} b_{2i+1} x^{2i+1}}{\sum_{i=0}^{\infty} a_{2i} x^{2i}} = -\begin{vmatrix} 0 & b_1 & b_3 & b_5 & b_7 & b_9 & \cdots \\ x^1 & 1 & a_2 & a_4 & a_6 & a_8 & \cdots \\ x^3 & 0 & 1 & a_2 & a_4 & a_6 & \cdots \\ x^5 & 0 & 0 & 1 & a_2 & a_4 & \cdots \\ x^7 & 0 & 0 & 0 & 1 & a_2 & \cdots \\ x^9 & 0 & 0 & 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{vmatrix} ; a_0 = 1$$

Link to Unit 4 and 64

Change the lobster:

$$p_n(x) = \sum_{i=0}^{n} A_i x^{n-i} = A_0 \prod_{j=1}^{n} (x - x_j)$$

Assigned basis determinant

$$\underline{D} = \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & \cdots \\ A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & \cdots \\ 0 & A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ 0 & 0 & A_0 & A_1 & A_2 & A_3 & \cdots \\ 0 & 0 & 0 & A_0 & A_1 & A_2 & \cdots \\ 0 & 0 & 0 & 0 & A_0 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix}$$

First modification
Delete in $\underline{D}$ the columns 2, 4, 6, … and push the remainder to the new determinant

$$\begin{array}{cccccc} \underline{\Delta}_1 & \underline{\Delta}_2 & \underline{\Delta}_3 & \underline{\Delta}_4 & \underline{\Delta}_5 & \underline{\Delta}_6 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

$$\underline{\Delta} = \begin{vmatrix} A_1 & A_3 & A_5 & A_7 & A_9 & A_{11} & \cdots \\ A_0 & A_2 & A_4 & A_6 & A_8 & A_{10} & \cdots \\ 0 & A_1 & A_3 & A_5 & A_7 & A_9 & \cdots \\ 0 & A_0 & A_2 & A_4 & A_6 & A_8 & \cdots \\ 0 & 0 & A_1 & A_3 & A_5 & A_7 & \ddots \\ 0 & 0 & A_0 & A_2 & A_4 & A_6 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{vmatrix}$$

$\underline{\Delta}$ contains information about the position of the n zeros x_j of $p_n(x)$ in the Gauss plane:
Iff all elements of the sequence

$$\underline{\Delta}_0 = A_0,\ \underline{\Delta}_1 = A_1,\ \underline{\Delta}_2 = \begin{vmatrix} A_1 & A_3 \\ A_0 & A_2 \end{vmatrix},$$

$$\underline{\Delta}_3=\begin{vmatrix} A_1 & A_3 & A_5 \\ A_0 & A_2 & A_4 \\ 0 & A_1 & A_3 \end{vmatrix}, \underline{\Delta}_4=\begin{vmatrix} A_1 & A_3 & A_5 & A_7 \\ A_0 & A_2 & A_4 & A_6 \\ 0 & A_1 & A_3 & A_5 \\ 0 & A_0 & A_2 & A_4 \end{vmatrix},\ldots,$$

$\underline{\Delta}_n=A_n*\underline{\Delta}_{n-1}$

are positive

then all x_j have negative real parts, e.g. are located in the left half-space of the Gauss plane,

$(\forall j)\ 90°<\arg(x_j)<270°$

(Routh-Hurwitz Theorem [1,2,3]).

Example

$p_3(x) = x^3+12x^2+54x+68$

$A_0=1,\ A_1=12,\ A_2=54,\ A_3=68$

$x_1=\mathbf{-2},\ x_{2,3}=\mathbf{-5}\pm 3j$

$$\Delta=\begin{vmatrix} 12 & 68 & 0 & 0 & \cdots \\ 1 & 54 & 0 & 0 & \cdots \\ 0 & 12 & 68 & 0 & \cdots \\ 0 & 1 & 54 & 0 & \cdots \\ \vdots & & & & \end{vmatrix} \rightarrow$$

$\Delta_0=1,\ \Delta_1=12,\ \Delta_2=580,\ \Delta_3=68*580$

$(\forall i)\Delta_i$ positive => $(\forall j)\mathrm{Re}(x_j)$ negative

Second and third modification

Find analogues to the first modification for the

segments
** $(\forall j)\ 90°+30°=120°<\arg(x_j)<270°-30°=240°$
** $(\forall j)\ 90°+45°=135°<\arg(x_j)<270°-45°=225°$
** $(\forall j)\ 90°+60°=150°<\arg(x_j)<270°-60°=210°$.
Find analogues to the first modifikation for the unit circle
** $(\forall j)\ |x_j|< 1$.
Find more modifications.
Discuss algebraic-geometric connections between coefficient determinant structures and related positions of the zeros of $p_n(x)$ in the Gauss plane, f.e. via conformal map [4,5].
Discuss the special case
„Determination/calculation of
** type/value (multiple, rational, …)
** inside area in the Gauss plane (vertical/horizontal
strip, rectangle, sector, circle, circular ring, …)
of the roots x_j of $p_n(x)$ only by coefficient-a_i-determinants“
from the point of view of
complex analysis, conformal map
information theory.

Links to Unit 4, 6.2, 10, 64, and 136.
References
[1] Weisstein, Eric W. "Routh-Hurwitz Theorem." From *MathWorld*-

-A Wolfram Web Resource.
http://mathworld.wolfram.com/Routh-HurwitzTheorem.html
[2] https://de.wikipedia.org/wiki/Hurwitzpolynom
[3] F.R. Gantmacher. Matrizenrechnung II, Kap.XV
[4] Weisstein, Eric W. "Conformal Mapping." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/ConformalMapping.html
[5] https://en.wikipedia.org/wiki/Conformal_map
[6] https://www.spektrum.de/news/inspiration-zwei-algorithmen-fuer-geniale-ideen/1828738

To be continued.
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